The Journal of Risk Management and Insurance Vol. 26 No. 1 (2022)

Life-cycle Consumption in the Presence of a Term Life Insurance, Voluntary, Altruistic Bequest, and Uncertain Life Span

Giuseppe Di Liddo

University of Bari, Bari, Italy
giuseppe.diliddo@uniba.it

Fabrizio Striani

University of Salento, Lecce, Italy
fabrizio.striani@unisalento.it

Received: January 14, 2022
Revised: April 21, 2022
Accepted: April 30, 2022

Abstract

In this paper, we extend the inter-temporal consumption model of Yaari in an asymmetric framework with market incompleteness, where agents can only take a long position in life insurance. We study consumption and saving behaviour in the presence of voluntary bequest concerns and term life insurance that can not be resold in the context of life expectancy uncertainty. We provide general insights that do not depend on specific forms of the utility function and risk aversion hypotheses. In particular, our results suggest that such policy leads to variable optimal consumption over time that is still affected by lifetime uncertainty.

Keywords: Life-cycle Model, Term Life Insurance, Bequest, Longevity Risk

1 Introduction

Economic theories of inter-temporal consumption aim to explain consumption and saving concerning economic agents' preferences throughout their lives. Modern inter-temporal consumption theory has origin in the 1950s, with models built on discounted utility theory that approached the question of inter-temporal consumption as a lifetime income optimization problem. Assuming that individuals are rational, perfectly know their lifetime, and have access to complete markets, Fisher (1930), Modigliani and Brumberg (1954) and Friedman (1957) developed what became known as the life-cycle models (LCMs), based on the idea that consumption profiles are set taking into account some "average" lifetime income (instead of income at any given age) that corresponds to a constant consumption profile. The specific conclusion of such models is that, in the first years of their lives, young individuals borrow to consume more than their income. Next, as their income rises through the years, their consumption rises slowly, and they begin to save more. During retirement, old individuals live off their savings.

The LCM of savings and consumption is at the core of most multi-period asset pricing and allocation models and the foundation of microeconomic consumer behaviour. The LCM original formulation assumed a deterministic lifetime horizon (Modigliani & Brumberg, 1954), starting from Yaari (1965) the analysis has been extended introducing lifetime uncertainty. In particular, Yaari (1965) assumes a concave utility function, stochastic lifetime, and a deterministic force of mortality with the entire survival curve known at time zero. In this framework, individuals face the problem of longevity and brevity risks using the actuarial note, leading the problem to the consumption smoothing whereby rational individuals seek to minimize disruptions to their standard of living over their entire life. We aim to extend the above-cited analyses in a more realistic setting characterized by market asymmetry in the following analysis. In particular, our research question is related to the consumption pattern of economic agents who can take a long position in life insurance, but not a short one, as in many fundamental world life insurance policies. We investigate consumer behaviour in a Yaari-like model to answer this question, which adds life insurance and life annuities to the inter-temporal consumption model. The asymmetry hypothesis introduces market incompleteness into the Yaari model since annuity markets are effectively shut.

Our results suggest that such policy leads to optimal variable consumption over time that is still affected by lifetime uncertainty. The reason is that common term life insurance cannot cover both brevity and longevity risk as the actuarial note does in Yaari (1965). Life insurance can at least cover against the longevity risk, and this situation may affect the way individuals look at the presence of bequest in their consumption plan. The rest of the paper is organized as follows: section 2 provides a literature review that introduces the ideas of smooth and bumped consumption profile in theoretical LCMs - based on different ex ante assumptions - in the light of some empirical works that observe such phenomenon in accurate data, section 3 illustrates our theoretical framework, section 4 provides some final remarks for further research on the topic.

2 Inter-temporal consumption and mortality uncertainty

The first example of mortality uncertainty in inter-temporal consumption models can be found in Yaari (1965), who solves the optimal portfolio problem employing actuarial notes that can be bought or sold by the consumer and cancelled upon the consumer's death. Indeed, a consumer who buys an actuarial note is buying an annuity that stipulates payments to the consumer during life at a rate higher than the interest rate. Upon the consumer's death, the insurance company has no further obligations to the consumer's estate. Reversely, a consumer who sells an actuarial note is getting a life-insured loan. Yaari (1965) shows that households have the incentive to insure against the loss of life entirely. He thus reaches the striking result that, with actuarially adequate life insurance, the death hazard drops out of the consumption Euler equation altogether. As a result, individuals are finally characterized by a smooth consumption profile setting a specific consumption plan that is continuous and equates marginal utility at all points (Heijdra, 2009, p. 609).

The subsequent theoretical literature on inter-temporal consumption and saving has expanded the framework of the analysis started by Yaari (1965) treating uncertainty under the form of deterministic forces of mortality (for example, the Gompertz-Makeham (GM) mortality law). Most of the results obtained suggest constant consumption profiles or, at least, "smooth" consumption profiles. However, the empirical literature (Alessie & Ree, 2009) suggests that real consumption per household seems to track income over the life cycle. Some theoretical investigations (Butler, 2001) have found explanations to such phenomenon, maintaining the hypothesis of the actuarial note as main insurance instrument used by consumers, expanding the set of factors that affects consumption in theoretical models, and/or using particular specifications of the utility function.

Despite these empirical results, starting from Yaari (1965), LCM models have been based on deterministic forces of mortality. For example, Merton (1971), modelling uncertainty of life expectancy under a deterministic force and, assuming the possibility of default on (formerly) risk-free assets, derives portfolio consumption and portfolio rules of alternative asset price dynamics, which changes are neither stationary nor independent. Richard (1975) provides a continuous-time model for optimal consumption, portfolio, and life insurance rules for an investor with an arbitrary but known lifetime distribution, generalizing the model by Merton (1971). The results of such generalization suggest that investors are likely to have "human capital" component of wealth, independent of their preferences and risky market opportunities, representing the certainty equivalent of their future net (wage) earnings. In addition, Richard (1975) finds explicit solutions, which are linear in wealth, for the investor characterized by constant relative risk aversion (CRRA) function. Levhari and Mirman (1977) investigate the effect of lifetime uncertainty on optimal consumption decisions focusing on changes in the distribution of lifetime uncertainty. For risk-averse individuals, such changes decrease consumption due to the higher probability of having a longer life and increase consumption due to the desire for sure consumption in the present. The stronger of these effects determines the effect of lifetime uncertainty on optimal consumption decisions. The significant result of Levhari and Mirman (1977) is that, if the utility function is Cobb-Douglas and the rate of return is not too large relative to the amount of future discounting, then lifetime uncertainty will always *increase* consumption.

Davies (1981) attempts to explain whether the continued accumulation, or mild dissaving, observed among retired people at that time, could be due by uncertain lifetime. He finds that, in the absence of annuities, after an initial period influenced by borrowing constraints, under CRRA, uncertain lifetime decreases consumption by a proportion increasing with age (if the elasticity of inter-temporal substitution in consumption is sufficiently tiny). The consequence is that if the reduction in consumption is significant enough, it can explain much of the lack of decumulation by the elderly.

Similar research on the elderly has been conducted by Kingston and Thorp (2005), trying to find new explanations for the well-known reluctance of retirees to buy life annuities (Milevsky & Huang, 2018). Their results suggest that, since the decision to purchase longevity insurance is mainly irreversible, a real option to delay annualization (RODA) generally has value in uncertain environments. They provide RODA analysis to the case of hyperbolic absolute risk aversion (HARA) preferences, the most straightforward representation of consumption habit. They find that the formula for the optimal timing of annuitization is surprisingly simple but yields only a myopic solution. The precise date of annuitization cannot be ascertained in advance.

Butler (2001) extends the analysis framework through a simple neoclassical life-cycle model in continuous time, in which the effects of endogenous labour supply, uncertain lifetime, and family composition on consumption and income profiles are jointly analysed. His model can generate a hump in the consumption profile and a co-movement of consumption and income during working life without relying on borrowing constraints.

The idea of a life-cycle consumption profile with a hump due to uncertain lifespan is still present in Feigenbaum (2008), who finds that the elasticity of inter-temporal substitution is close to that estimated in the buffer-stock saving model by Gourinchas and Parker (2002), where borrowing constraints primarily account for the consumption hump. Since borrowing is virtually eliminated in the model with brevity risk (mortality risk), *mortality supplants the borrowing constraint* as the explanation for the hump with these parameters. If a pay-as-you-go social security system is also incorporated in the model, brevity risk can no longer account for the observed properties of the hump.

Dybvig and Liu (2010) use a framework characterized by uncertain lifetime with Poisson arrival of mortality at a fixed hazard rate to show that effect of retirement flexibility and inability to borrow against future labour income can significantly affect optimal consumption and investment. With voluntary retirement, an optimal wealth-to-wage ratio threshold for retirement and human capital correlates negatively with the stock market, even when wages have zero or slightly positive market risk exposure. Consequently, investors optimally invest more in the stock market than without retirement flexibility. Both consumption and portfolio choice jump at the endogenous retirement date. The inability to borrow limits hedging and reduces the value of labour income, the wealth-to-wage ratio threshold for retirement, and the stock investment.

Lachance (2012) studies the conditions for optimal individual retirement savings strategies within a life-cycle framework implementing the the model of Yaari (1965) with an uncertain lifetime and borrowing constraints, providing solutions both for the general case and for cases leading to closed-form equations such as power utility and Gompertz mortality. Illustrations for a wide range of parameters indicate that starting to save for retirement in the first phase of one's career is rarely optimal.

Chen and Lau (2016) use an overlapping-generations (OLG) model with endogenous retirement and saving to study the trade-off between saving and retirement age in response to mortality decline. They find that when life expectancy increases by one year, people delay retirement by about four months. Furthermore, the percentage of lifetime spent in working decreases, and people have to save more for post retirement years.

Recent examples of lifetime uncertain in LCMs are Bloom, Canning, and Moore (2014), Chen and Lau (2016), Huang, Milevsky, and Salisbury (2017), Milevsky and Huang (2018), and Mao, Ostaszewski, and Wen (2019), that further expands the analyses introducing new hypotheses, while maintaining the main idea of a certain distribution of life expectancy perceived by economic agents. To the best of our knowledge, a relevant exception is provided by Huang, Milevsky, and Salisbury (2012) that extend the Yaari (1965) LCM of consumption allowing for *stochastic* mortality. In this case consumers can adapt their consumption strategy to new information about their health status and mortality probability as it becomes available.

From the above review of the theoretical literature, we can note that the central tenet of the LCMs is that people work, earn income, and save (accumulate wealth) during the working period, and retire and dissave (decumulate wealth) when old. However, empirical studies, in general, show that the elderly continue to accumulate wealth (save) or that they decumulate their wealth (dissave) by a rate of wealth decumulation smaller than predicted by LCMs (Bodie, Detemple, Otruba, & Walter, 2004; Deaton, 1991). Furthermore, some empirical research shows that both bequest motives and precautionary saving are significant as explanations for the failure of the retired elderly to reduce their wealth as quickly as expected.

For example, De Nardi, French, and Jones (2010) estimate a model of saving for retired single people, that includes heterogeneity in medical expenses and life expectancies and bequest motives, finding that out-of-pocket medical expenses rise quickly with age and permanent income. They also found that longevity risk (the risk of living long and requiring expensive medical care) is a key driver of saving for many higher income elderly.

Niimi and Horioka (2019) analyse the determinants of the wealth decumulation behaviour of the retired elderly in Japan using unique information from two household surveys. Their results suggest the possibility that the financial burden of parental care may also affect the wealth decumulation behaviour of the retired elderly in Japan. Given that parental care responsibilities tend to arise relatively late in life, often after retirement, in the case of Japan, the financial burden of parental care may be a relevant issue when analysing the wealth decumulation behaviour of the elderly. Their results highlight the relative importance of precautionary saving and bequest motives in explaining the lower than expected rates of wealth decumulation of the retired elderly. Such importance is well recognized in the empirical literature (see Niimi and Horioka (2019)) and suggests that the LCM theoretical model should include the presence of charitable bequest to depict the framework of an intertemporal consumption choice.

The following section provides an LCM general framework that could be useful to find irregularities in the consumption profile under lifespan uncertainty, without recurring to a particular functional form of the utility function. We assume that consumers can subscribe, a term life insurance policy that differently to Yaari (1965), can not be resold. Such policy is used to cover individuals' longevity risk. In general, such risk arises when uncertainty around an individual's life span causes them to underestimate how long they will live for and are likely to erode all their wealth before the last years of their life. In our model, this problem due of the longevity risk is connected with the risk of not leaving a positive bequest to their heirs.

Furthermore, we impose a constraint of no indebtedness, one of the main novelties of our model. Such constraint has been introduced following some intuitions provided by the empirical literature on elderly saving, which has shown that older people continue to save a lot for some reason. Such a reason could be the old weight agents assigned to the bequest for their heirs. Consequently, we also include the bequest in the inter-temporal utility function. Following the previous theoretical literature, we continue to assume uncertainty in the lifespan, but we do not express uncertainty in a particular form to maintain our model the more general as possible.

3 LCM generalization

In this section we further extend the LCM model of Yaari (1965), highlighting the implications of the model with regard to two types of issues: the first is related to the presence of an altruistic bequest (Lord & Rangazas, 1991), the second to the fact that the market is no longer complete in the presence of term life policy instead of actuarial note. The last assumption reproduces many real world life insurance policies. In the first subsection we present the baseline model with certain lifespan, useful as benchmark for the extension to the case of lifetime uncertainty. In the second subsection we introduce uncertainty in the lifespan.

3.1 The baseline model in the presence of certain lifespan

We consider a consumer characterized by a certain lifespan $n \geq 1$. At time $t \in [0, n-1]$ he or she faces a problem of inter-temporal choice between saving and consumption. Assuming that consumer's preferences on consumption are additively separable, following Samuelson (1969), we can write the consumer's lifetime utility function U_t at time t as:

$$U_t = \sum_{i=0}^{n-1-t} \left(\frac{1}{1+\delta} \right)^i u\left(c_{t+i}\right). \tag{1}$$

In (1), c is the instantaneous consumption, δ represents the rate of inter-temporal preferences and $u(c)$ is the instantaneous utility of consumption, increasing and concave. Supposing to have m financial assets, we assume uncertainty on future incomes expressing the consumer's budget constraint as follows:

$$\sum_{j=1}^{m} A_{t+i}^{j} = \sum_{j=1}^{m} \left(1 + r_{t+i}^{j}\right) A_{t+i-1}^{j} + x_{t+i} - c_{t+i}. \tag{2}$$

In (2), A_{t+i-1}^{j} represents the amount invested in the $j-th$ activities; r_{t+i}^{j} is the rate of return of financial assets; x_{t+i} is the labour income, and c_{t+i} is the consumption. The representative consumer is uncertain over the future labour and capital incomes. It follows that the consumer objective function can be expressed (Hall, 1989) in terms of expected lifetime utility:

$$U_t = E_t \left[\sum_{i=0}^{n-1-t} \left(\frac{1}{1+\delta} \right)^i u\left(c_{t+i}\right) \right]. \tag{3}$$

Let $\sum_{j=1}^{m} A_{t+i}^{j} = W_{t+i}$ and r^w the rate of return referred to wealth. Letting $R^w = 1 + r^w$ the gross rate of return of wealth, we can express the budget constraint (2) in terms of wealth W:

$$W_{t+i} = R_{t+i}^{w} W_{t+i-1} + x_{t+i} - c_{t+i}. \tag{4}$$

Assuming that the wealth at time $t-1$ is known at time t, i.e. $W_{t-1} = W^0$, we consider, without loss of generality, only two periods. The budget constraint (4) becomes:

$$W_t = R_t^w W^0 + x_t - c_t, \tag{5}$$

$$W_{t+1} = R_{t+1}^{w} W_t + x_{t+1} - c_{t+1}. \tag{6}$$

Substituting (5) into (6) we have:

$$W_{t+1} = R^w_{t+1} R^w_t W^0 + R^w_{t+1} (x_t - c_t) + x_{t+1} - c_{t+1}. \tag{7}$$

The consumer's choice problem over the two periods can now be expressed as follows:

$$\max_{c_t, c_{t+1}} u(c_t) + \frac{1}{1+\delta} \sum_{s=1}^{z} \pi(s) u(c^s_{t+1}), \tag{8}$$

subject to equation (7).

In (8), $\pi(s)$ represents the probability that $s \in \{1, \ldots, z\}$ is one of the states of nature. We derive the first order conditions (FOCs) of the problem using the Lagrangian:

$$\begin{aligned} \mathcal{L} = {} & u(c_t) + \frac{1}{1+\delta} \sum_{s=1}^{z} \pi(s) u(c^s_{t+1}) \\ & - \sum_{s=1}^{z} \lambda^s_{t+1} \left(W^s_{t+1} - R^w_{t+1}(s) R^w_t W^0 - R^w_{t+1}(s) (x_t - c_t) - (x^s_{t+1} - c^s_{t+1}) \right) \end{aligned} \tag{9}$$

Maximization of (9) leads to the following FOCs of the problem:

$$\begin{cases} u'(c_t) = \sum_{s=1}^{z} \lambda^s_{t+1} R^w_{t+1}(s) \\ \dfrac{1}{1+\delta} \pi(s) u'(c^s_{t+1}) = \lambda^s_{t+1}, \end{cases} \tag{10}$$

where λ^s_{t+1} is the Lagrangian multiplier.

From (10) it follows that:

$$u'(c_t) = \sum_{s=1}^{z} \left(\frac{1}{1+\delta} \right) \pi(s) u'\left(c^s_{t+1}\right) R^w_{t+1}(s).$$

That is,

$$u'(c_t) = \frac{1}{1+\delta} E_t \left[u'(c_{t+1}) R^w_{t+1} \right]. \tag{11}$$

The right hand side of (11) represents the condition for the optimal consumption at time t and takes the form of the Euler equation.

In order to analyse the solution of the problem, we can recur to some concepts of the portfolio theory. In particular, we introduce portfolio shares $\alpha^j_t = \frac{A^j_t}{W_t}$, where:

$$\sum_{j=1}^{m} A^j_{t+i} = W_{t+i} \Rightarrow \sum_{j=1}^{m} \frac{A^j_{t+i}}{W_{t+i}} = 1 \Rightarrow \sum_{j=1}^{m} \alpha^j_{t+i} = 1. \tag{12}$$

Note that, given (4), we have:

$$c_{t+i} = W_{t+i-1} \sum_{j=1}^{m} R^j_{t+i} \alpha^j_{t+i-1} + x_{t+i} - W_{t+i}. \tag{13}$$

Substituting (13) in the objective function (3), we obtain:

$$U_t = E_t \left[\sum_{i=0}^{n-1-t} \left(\frac{1}{1+\delta} \right)^i u \left(W_{t+i-1} \sum_{j=1}^{m} R^j_{t+i} \alpha^j_{t+i-1} + x_{t+i} - W_{t+i} \right) \right]. \tag{14}$$

Considering two periods, we get the problem:

$$\max_{\alpha^j_t} u \left(W_{t-1} \sum_{j=1}^{m} R^j_t \alpha^j_{t-1} + x_t - W_t \right) + \left(\frac{1}{1+\delta} \right) E_t \left[u \left(W_t \sum_{j=1}^{m} R^j_{t+1} \alpha^j_t + x_{t+1} - W_{t+1} \right) \right] \tag{15}$$

subject to

$$\sum_{j=1}^{m} \alpha^j_t = 1. \tag{16}$$

The FOCs of (15) subject to (16) are:

$$\frac{1}{1+\delta} \sum_{s=1}^{z} \pi(s) \, u'\left(c^s_{t+1}\right) W_t R^j_{t+1}(s) = \sum_{s=1}^{z} \nu^s_{t+1}, \qquad j = 1, ..., m, \tag{17}$$

where $\sum_{s=1}^{z} \nu^s_{t+1}$ is the set of the Lagrangian multiplier for each state of nature. Therefore, we can express (17) in the form:

$$\frac{1}{1+\delta} W_t E_t \left[u'(c_{t+1}) R^j_{t+1} \right] = \sum_{s=1}^{z} \nu^s_{t+1}, \qquad j = 1, ..., m. \tag{18}$$

Suppose we have, next to m financial assets considered until now, another asset without risk $R^0 = 1 + r^0$. That is, there are $m+1$ titles with $\sum_{j=0}^{m} \alpha^j_t = 1$.

It follows that the above maximization (15) remains mostly unchanged, with the addition of a further FOC:

$$\left(\frac{1}{1+\delta} \right) W_t E_t \left[u'(c_{t+1}) R^0 \right] = \sum_{s=1}^{z} \nu^s_{t+1}. \tag{19}$$

From (17) and (19) we get:

$$E_t \left[u'(c_{t+1}) \left(R^j_{t+1} - R^0 \right) \right] = 0. \tag{20}$$

It follows that the optimum portfolio described by (20) is composed by the risk-free asset title and the composite risky asset. The latter is the optimum combination of all risky securities that is known in the literature as "mutual fund". Tobin (1958) shows that, under certain conditions, the agent decisions of portfolio allocation may be regarded as a two-stage process with separate choices on the optimum allocation ratio within the risky assets and the optimum allocation ratio between the risk-free asset and the whole risky assets.

Cass and Stiglitz (1970) show the condition for an optimal mutual fund, i.e. the separation of risky assets and risk-free activity. Such separation occurs under specific hypotheses on the probability distribution of returns and the utility function. Yields that follow a normal distribution satisfy the requirements on probability distribution, while the utility function must include a particular form of risk aversion. The classic examples of such functions are the quadratic utility function, the exponential utility function, the logarithmic utility function, and the CRRA utility function.

Therefore, introducing one of these explicit utility functions in (17) and (19), we can derive the optimal proportion of portfolio shares in the chosen portfolio. That is, the optimal portfolio composition that maximizes the individual inter-temporal utility function over the two period considered.

3.2 The model in the presence of uncertain lifespan

We introduce uncertainty in the baseline model by means of Q_t, the chance of being alive at the time t. The presence of uncertainty on lifetime leads to some issues: the first is related to the fact that the market is no longer complete since now the array of returns no longer has full rank; the second is related to the bequest (B). We assumes an altruistic bequest, i.e. individuals care about future generations, according to Cocco (2005). According to Deelstra, Devolder, and Melis (2021), it follows that the utility function includes both bequest and consumption as arguments.

In order to cope with these problems and restore completeness, a further constraint is necessary. Let's start by B_t. Given its presence, (3) now takes the form:

$$U_t = E_t \left[\sum_{i=0}^{n-1-t} \left(\frac{1}{1+\delta} \right)^i \left(u\left(c_{t+i}\right) Q_{t+i} + u\left(B_{t+i}\right)\left(1 - Q_{t+i}\right)\right) \right]. \tag{21}$$

That is,

$$U_t = E_t \left[\sum_{i=0}^{n-1-t} \gamma_{t+i} u\left(c_{t+i}\right) + \sum_{i=0}^{n-1-t} \eta_{t+i} u\left(B_{t+i}\right) \right], \tag{22}$$

where:

$$\gamma_{t+i} = \left(\frac{1}{1+\delta} \right)^i Q_{t+i}, \tag{23}$$

and

$$\eta_{t+i} = \left(\frac{1}{1+\delta}\right)^i (1 - Q_{t+i}). \tag{24}$$

The term γ_{t+i} in (23) can be viewed as a 'modified' factor of inter-temporal preferences (weighted for the probability to be alive in the next period) and it is lower than the term $\left(1/(1+\delta)\right)^i$because $Q_{t+i} \leq 1$. The same holds for η_{t+i} in (24), which is weighted for the probability to be dead in the next period.

We now introduce the possibility to subscribe a term life insurance policy. Following Rojeck (2019), we assume that the insurance policy is the term life insurance policy in the form:

$$V = \sum_{k=1}^{n-1-t} p\left(Q_{t+k-1} - Q_{t+k}\right)(R)^k \qquad with \quad Q_{t+k} < Q_{t+k-1}, \tag{25}$$

where $(Q_{t+k-1} - Q_{t+k})$ is the probability of no longer being alive in a given period, p is the fixed premium paid by the consumer to the company insurance ($p \geq 0$); and R is the gross rate of return which guarantees technical equity.

Assuming that R, as usually happens in insurance companies, is the gross rate of return risk-free (R^0), it follows that:

$$V = pF\left(R^0\right), \tag{26}$$

where the term $F(R^0)$ represents the revaluation offer of the insurance company such that $F(R^0) = \sum_{k=1}^{n-1-t} (Q_{t+k-1} - Q_{t+k}) \left(R^0\right)^k$.

Now we can distinguish two scenarios: in the first one the consumer does not buy life insurance, in the second one he or she decides to make such a purchase.

3.2.1 First scenario: no life insurance

In this scenario, the agent does not know exactly the duration of his/her life but eventual debt must be extinguished before death. That is, he or she imposes a constraint of no indebtedness, one of the main novelty of our model.

As before we consider only two periods, so $i = (0, 1)$. It follows that the bequest will be:

$$B_t = W_t; \qquad B_{t+1} = W_{t+1} \tag{27}$$

Now the individual inter-temporal maximization problem can be expressed as follows:

$$\max_{c_t, c_{t+1}, W_t, \alpha_t^j} \gamma_t u(c_t) + \eta_t u(W_t) + \sum_{s=1}^{z} \pi(s)\left(\gamma_{t+1} u(c_{t+1}) + \eta_{t+1} u(W_{t+1})\right) \tag{28}$$

subject to

$$W_t = R_t^w W^0 + x_t - c_t, \tag{29a}$$
$$W_{t+1} = R_{t+1}^w W_t + x_{t+1} - c_{t+1}, \tag{29b}$$
$$W_{t+1} > 0, \tag{29c}$$
$$W_t > 0, \tag{29d}$$
$$\sum_{j=1}^{m} \alpha_t^j = 1. \tag{29e}$$

Note that the debt constraint should be $W_{t+1} \geq 0$, instead we will find the solutions only for $W_{t+1} > 0$ avoiding the less interesting cases where $W_{t+1} = 0$ that could lead to corner solutions of the problem. Note also that the constraint $W_{t+1} > 0$ restores completeness.

Comparing (28) to (15), note that the presence of uncertainty on lifespan, captured by the terms γ_{t+i} and η_{t+i}, worsens the inter-temporal utility profile. Uncertainty leads to a lower level of utility, despite maximization process and the following optimal conditions remains qualitatively the same.

The Lagrangian of the problem (28) subject to (29) is:

$$\begin{aligned} \mathcal{L} \; &= \gamma_t u(c_t) + \eta_t u(W_t) + \sum_{s=1}^{z} \pi(s)\left(\gamma_{t+1} u(c_{t+1}) + \eta_{t+1} u(W_{t+1})\right) \\ &- \lambda_t \left(W_t - W^0 R_t^w - x_t + c_t\right) - \sum_{s=1}^{z} \lambda_{t+1}^s \left(W_{t+1}^s - W_t R_{t+1}^w(s) - x_{t+1}^s + c_{t+1}^s\right) \\ &+ \mu_t W_t + \sum_{s=1}^{z} \mu_{t+1}^s W_{t+1}^s + \sum_{s=1}^{z} \nu_{t+1}^s \left(1 - \sum_{j=1}^{m} \alpha_t^j\right) \end{aligned} \tag{30}$$

From (30) it follows that the FOCs of problem (28) subject to (29) are:

$$\begin{cases} \gamma_t u'(c_t) = \lambda_t \\ \pi(s)\gamma_{t+1} u'\left(c_{t+1}^s\right) = \lambda_{t+1}^s \\ \eta_t u'(W_t) + \sum_{s=1}^{z} \lambda_{t+1}^s R_{t+1}^w(s) + \mu_t = \lambda_t \\ \sum_{s=1}^{z} \lambda_{t+1}^s W_t R_{t+1}^j(s) = \sum_{s=1}^{z} \nu_{t+1}^s \qquad \textit{with } j = 1, ..., m. \end{cases} \tag{31}$$

From the first three FOCs in (31) we have:

$$\begin{aligned} &\gamma_t u'(c_t) - \eta_t u'(W_t) = E_t\left[\gamma_{t+1} u'(c_{t+1}) R_{t+1}^w\right] \Rightarrow \\ &\Rightarrow u'(c_t) = E_t\left[\frac{\gamma_{t+1}}{\gamma_t} u'(c_{t+1}) R_{t+1}^w\right] + \frac{\eta_t}{\gamma_t} u'(W_t) \end{aligned} \tag{32}$$

Let's explore the portfolio choice. From the fourth FOC in (31), after few computations, we have:

$$W_t E_t \left[\gamma_{t+1} u'\left(c_{t+1}\right) R^j_{t+1}\right] = \sum_{s=1}^{z} \nu^s_{t+1} \qquad with\ j = 1, ..., m \tag{33}$$

Introducing the risk free assets as in the baseline model, we get:

$$W_t E_t \left[\gamma_{t+1} u'\left(c_{t+1}\right) R^0\right] = \sum_{s=1}^{z} \nu^s_{t+1}. \tag{34}$$

From (33) and (34) we obtain

$$E_t \left[\gamma_{t+1} u'\left(c_{t+1}\right) \left(R^j_{t+1} - R^0\right)\right] = 0 \tag{35}$$

Condition (35) is very similar to (20), the only difference is the presence of the intertemporal preference rate changes. Furthermore, considering that the no debt constraint $W^s_{t+1} > 0$ leads to restoring the full rank in the returns matrix, we can find an explicit solution of the problem using explicit utility functions as in the case with no lifetime uncertainty in section 3.1.

Comparing (32) to (11), note that now the choice on consumption profile depends also on the bequest $W_t = B_t$. The presence of B_t makes the consumption profile smaller than the one obtained in the certainty case because, for altruistic reason, individuals must save part of their wealth. In order to solve this problem, Yaari (1965) introduces the actuarial note. Instead, in the next section, we will introduce the term life insurance in the form described at the beginning of the subsection 3.2.

3.2.2 Second scenario: life insurance

Here we introduce in the first scenario the life insurance policy described by (25-26). The budget constraint becomes:

$$W_{t+i} = R^w_{t+i} W_{t+i-1} + x_{t+i} - c_{t+i} - p. \tag{36}$$

Considering two periods, the bequest is given by the sum of wealth and the amount paid by the insurance policy in case of death:

$$B_t = V + W_t \tag{37a}$$
$$B_{t+1} = V + W_{t+1} \tag{37b}$$

Considering (37a), (37b), and (26), the maximization problem (28) takes now the form:

$$\max_{p, c_t, c_{t+1}, W_t, \alpha^j_t} \gamma_t u\left(c_t\right) + \eta_t u\left(pF(R^0) + W_t\right) + E_t \left[\gamma_{t+1} u\left(c_{t+1}\right) + \eta_{t+1} u\left(pF(R^0) + W_{t+1}\right)\right], \tag{38}$$

subject to:

$$W_t = R_t^w W^0 + x_t - c_t - p, \tag{39a}$$
$$W_{t+1} = R_{t+1}^w W_t + x_{t+1} - c_{t+1} - p, \tag{39b}$$
$$pF(R^0) + W_{t+1} > 0, \tag{39c}$$
$$pF(R^0) + W_t > 0, \tag{39d}$$
$$\sum_{j=1}^{m} \alpha_t^j = 1. \tag{39e}$$

The Lagrangian of problem (38) subject to (39) is:

$$\begin{aligned}\mathcal{L} = {} & \gamma_t u(c_t) + \eta_t u(pF(R^0) + W_t) + \sum_{s=1}^{z} \pi(s)(\gamma_{t+1} u(c_{t+1}^s) + \eta_{t+1} u(pF(R^0) + W_{t+1}^s)) \\ & - \lambda_t \left(W_t - W^0 R_t^w - x_t + c_t + p\right) - \sum_{s=1}^{z} \lambda_{t+1}^s \left(W_{t+1}^s - W_t R_{t+1}^w(s) - x_{t+1}^s + c_{t+1}^s + p\right) \\ & + \mu_t \left(pF(R^0) + W_t\right) + \sum_{s=1}^{z} \mu_{t+1}^s \left(pF(R^0) + W_{t+1}^s\right) \\ & + \sum_{s=1}^{z} \nu_{t+1}^s \left(1 - \sum_{j=1}^{m} \alpha_t^j\right).\end{aligned} \tag{40}$$

(40) leads to the FOCs:

$$\begin{cases} F\left(R^0\right)\left(\eta_t u'\left(pF(R^0) + W_t\right) + \sum_{s=1}^{z} \pi(s)\, \eta_{t+1} u'\left(pF(R^0) + W_{t+1}^s\right) + \mu_t + \sum_{s=1}^{z} \mu_{t+1}^s\right) = \Lambda \\ \gamma_t u'(c_t) = \lambda_t \\ \pi(s)\, \gamma_{t+1} u'\left(c_{t+1}^s\right) = \lambda_{t+1}^s \\ \eta_t u'\left(pF(R^0) + W_t\right) + \sum_{s=1}^{z} \lambda_{t+1}^s R_{t+1}^w(s) + \mu_t = \lambda_t \\ \sum_{s=1}^{z} \lambda_{t+1}^s W_t R_{t+1}^j(s') = \sum_{s=1}^{z} \nu_{t+1}^s \qquad with\ j = 1, ..., m. \end{cases} \tag{41}$$

where $\Lambda = \lambda_t + \sum_{s=1}^{z} \lambda_{t+1}^s$. Using the first four FOCs in (41) we obtain:

$$\begin{aligned} u'(c_t) = \bar{R}^0 E_t \Bigg[& \frac{\gamma_{t+1}}{\gamma_t} u'(c_{t+1}) \left(R_{t+1}^w + \frac{1}{F(R^0)}\right) \\ & - \frac{\eta_{t+1}}{\gamma_t} u'\left(pF\left(R^0\right) + W_{t+1}\right)\Bigg], \end{aligned} \tag{42}$$

where $\bar{R}^0 = \frac{F(R^0)}{F(R^0)-1}$. Comparing (42) to (32), note that now the choice on consumption does not depends on the bequest $B_t = W_t$ since life insurance covers the so-called longevity risk (but not brevity risk) related to the possibility to die without leaving a positive bequest. It follows that the consumer is in a better situation than occurs in (32). By other words, due to the presence of the life insurance, the consumer has not to care about the bequest B_t, since, in case of death, his/her heirs would obtain the amount V from the insurance company. However, the consumer is still in a worst situation than (11), due to the presence of the policy prize and the terms γ_{t+1} and η_{t+1} that lower the consumption profile compared to (11). We proceed with the portfolio choice in this case. After few substitutions we can express the fifth FOC in (41) under the form of (34), and finally the condition (35). Note that, in the presence of common life insurances, the term γ_{t+1} is still present in the optimal portfolio condition.

3.3 Discussion

In our study we have modelled a LCM of saving and consumption, following the general framework provided by Yaari (1965), extending the analysis to some aspects of the LCM that have not been investigated so far. Regarding existing studies. Merton (1971) have introduced a particular form of uncertainty about life; Richard (1975), basing on Merton (1971), considers an arbitrary but known distribution of the duration of life and shows that agents' behaviour is identical to that obtained by Yaari (1965). Levhari and Mirman (1977) show that the uncertainty on the duration of life always produces an increase in the initial consumption, in this way they solve the problem of brevity but not of longevity risks, as instead it is demonstrated in our model.

Davies (1981), basing on Levhari and Mirman (1977) and Yaari (1965), shows that the consumption profile is decreasing with age, i.e., pensioners use resources slowly. All these studies do not consider the presence of a bequest in their analyses.

Butler (2001) does not consider the possibility of a borrowing constraint and thus generates a hump in consumption which is also verified by Feigenbaum (2008) and Gourinchas and Parker (2002)), the last also shows that wealth is accumulated only for precautionary reason. In our model, the presence of life insurance meets the needs arising from precautionary saving. Dybvig and Liu (2010) show that flexibility in retirement and the inability to borrow influence consumption by reducing it. As mentioned above, the presence of life insurance does not excessively reduce consumption. Lachance (2012) specifies that uncertain life expectancy and borrowing constraints point to the fact that it is rarely optimal to save for retirement in the early stage of a working career. In addition, Niimi and Horioka (2019) highlight that precautionary saving and the accidental bequest motive justify rates of wealth decumulation. This not happens in our case because we consider altruistic and voluntary bequest along with life insurance that eliminates this decumulation. In our case, the presence of bequest motivations leads to the opposite result.

Chen and Lau (2016) point out that people save more for retirement as their working life is shorter. Taking out life insurance avoids this. De Nardi et al. (2010) highlights that the bequest motive makes longevity a key saving factor for older people with higher incomes. Instead, in our model the longevity risk is covered by the insurance policy.

In our model, differently from previous studies, we extend the analysis of Yaari (1965), introducing uncertainty on the lifespan of the consumer in the presence of an altruistic bequest, the no-indebtedness constraint, and a term life insurance policy. Differently to Levhari and Mirman (1977) our results show that consumers solve the problem of longevity but not of brevity risks. Furthermore, we confirm the results obtained by Niimi and Horioka (2019) in the presence of altruistic bequest too. Differently to Chen and Lau (2016) and De Nardi et al. (2010), we obtain a different accumulation rate during retirement.

4 Final remarks

The presence of the actuarial note in the model of Yaari (1965) makes quite irrelevant uncertainty on the date of death since the actuarial note can be both purchased and sold in each period. Indeed, if we suppose that the agent is not limited to the sole subscription of the policy, but that he or she may also sell it at time t, he or she can increase the wealth available at time t and, in case of death at time $t+1$, he or she could preserve a positive bequest B_{t+1}. That is, actuarial note covers both longevity and brevity risk, eliminating the effect of uncertainty on lifespan in the choice of the consumption profile.

On the contrary, term life insurance can only be purchased in our model. The consequence is that, in the presence of life uncertainty, such insurance policy improves the consumption profile of consumers. Indeed, term life insurance can at least cover the longevity risk. It follows that despite the optimal consumption condition disappearing, the part concerning the bequest explicitly at the time t uncertainty still affects consumers' decisions.

As a consequence, in our model the term life insurance leads to a better situation if compared to the absence of insurance, but here consumption is not smooth as in the baseline model derived from Yaari (1965) and then individuals gain a lower utility if compared to the situation in the presence of the actuarial note. In addition, differently to Yaari (1965), in our framework consumers pay an insurance premium that negatively affects the wealth without the possibility to sell the insurance policy in the future, in case increasing wealth in a second moment.

In our framework, the presence of uncertainty in consumer choice, combined to the aim to maintain a positive bequest for their heirs, could be and explanation the continued accumulation, or mild dissaving, observed among retired people in the empirical literature (Davies, 1981; Kingston & Thorp, 2005; Milevsky & Huang, 2018). Future research will be conducted including particular forms of consumer utility in our model, in order to compare situations with different risk aversion degrees of economic agents.

References

Alessie, R., & Ree, J. (2009, March). Explaining The Hump In Life Cycle Consumption profiles. *De Economist*, *157*(1), 107-120. doi: 10.1007/s10645-009-9119-4

Bloom, D. E., Canning, D., & Moore, M. (2014). Optimal retirement with increasing longevity. *The Scandinavian Journal of Economics*, *116*(3), 838-858.

Bodie, Z., Detemple, J. B., Otruba, S., & Walter, S. (2004). Optimal consumption-portfolio choices and retirement planning. *Journal of Economic Dynamics and Control*, *28*(6), 1115-1148.

Butler, M. (2001). Neoclassical life-cycle consumption: a textbook example. *Economic Theory*, *17*(1), 209-221.

Cass, D., & Stiglitz, J. E. (1970). The structure of investor preferences and asset returns, and separability in portfolio allocation: A contribution to the pure theory of mutual funds. *Journal of Economic Theory*, *2*(2), 122-160.

Chen, Y., & Lau, S.-H. P. (2016). Mortality decline, retirement age, and aggregate savings. *Macroeconomic Dynamics*, *20*(3), 715-736. doi: 10.1017/S136510051400056X

Cocco, J. F. (2005). Consumption and portfolio choice over the life cycle. *Review of Financial Studies*, *18*(2), 491-533.

Davies, J. B. (1981). Uncertain Lifetime, Consumption, and Dissaving in Retirement. *Journal of Political Economy*, *89*(3), 561-577.

De Nardi, M., French, E., & Jones, J. B. (2010, February). Why Do the Elderly Save? The Role of Medical Expenses. *Journal of Political Economy*, *118*(1), 39-75. doi: 10.1086/651674

Deaton, A. (1991). Saving and Liquidity Constraints. *Econometrica*, *59*(5), 1221-1248.

Deelstra, G., Devolder, P., & Melis, R. (2021, June). Optimal annuitisation in a deterministic financial environment. *Decisions in Economics and Finance*, *44*(1), 161-175. Retrieved from `https://ideas.repec.org/a/spr/decfin/v44y2021i1d10.1007_s10203-020-00316-5.html` doi: 10.1007/s10203-020-00316-

Dybvig, P. H., & Liu, H. (2010). Lifetime consumption and investment: Retirement and constrained borrowing. *Journal of Economic Theory*, *145*(3), 885-907.

Feigenbaum, J. (2008). Can mortality risk explain the consumption hump? *Journal of Macroeconomics*, *30*(3), 844-872.

Fisher, I. (1930). *The theory of interest: As determined by impatience to spend income and opportunity to invest it.* Macmillan, New York.

Friedman, M. (1957). *A theory of the consumption function.* National Bureau of Economic Research, Inc.

Gourinchas, P.-O., & Parker, J. A. (2002). Consumption over the life cycle. *Econometrica*, *70*(1), 47-89. doi: 10.1111/1468-0262.00269

Hall, R. E. (1989). Consumption. In R. J. Barro (Ed.), *Modern business cycle theory* (chap. 4). Harvard University Press.

Heijdra, B. (2009). *Foundations of modern macroeconomics*. OUP Oxford.

Huang, H., Milevsky, M., & Salisbury, T. (2017). Retirement spending and biological age. *Journal of Economic Dynamics and Control*, *84*, 58 - 76.

Huang, H., Milevsky, M. A., & Salisbury, T. S. (2012). Optimal retirement consumption with a stochastic force of mortality. *Insurance: Mathematics and Economics*, *51*(2), 282 - 291. doi: https://doi.org/10.1016/j.insmatheco.2012.03.013

Kingston, G., & Thorp, S. (2005). Annuitization and asset allocation with HARA utility. *Journal of Pension Economics and Finance*, *4*(03), 225-248.

Lachance, M.-E. (2012). Optimal onset and exhaustion of retirement savings in a life-cycle model. *Journal of Pension Economics and Finance*, *11*(01), 21-52.

Levhari, D., & Mirman, L. J. (1977). Savings and Consumption with an Uncertain Horizon. *Journal of Political Economy*, *85*(2), 265-281.

Lord, W., & Rangazas, P. (1991). Savings and wealth in models with altruistic bequests. *The American Economic Review*, *81*(1), 289–296.

Mao, H., Ostaszewski, K. M., & Wen, Z. (2019). The retirement decision with consideration of part-time work after retirement. *Journal of Insurance Issues*, *42*(1), 86–112.

Merton, R. C. (1971). Optimum consumption and portfolio rules in a continuous-time model. *Journal of Economic Theory*, *3*(4), 373 - 413. doi: https://doi.org/10.1016/0022-0531(71)90038-X

Milevsky, M. A., & Huang, H. (2018). The utility value of longevity risk pooling: Analytic insights. *North American Actuarial Journal*, *22*(4), 574-590. doi: 10.1080/10920277.2018.1467271

Modigliani, F., & Brumberg, R. (1954). Utility analysis and the consumption function: An interpretation of cross-section data. *Post Keynesian economics*, *1*, 388–436.

Niimi, Y., & Horioka, C. Y. (2019). The wealth decumulation behavior of the retired elderly in japan: The relative importance of precautionary saving and bequest motives. *Journal of the Japanese and International Economies*, *51*, 52 - 63. doi: https://doi.org/10.1016/j.jjie.2018.10.002

Richard, S. F. (1975). Optimal consumption, portfolio and life insurance rules for an uncertain lived individual in a continuous time model. *Journal of Financial Economics*, *2*(2), 187 - 203. doi: https://doi.org/10.1016/0304-405X(75)90004-5

Rojeck, R. P. (2019). Life insurance. In *Wealth: The ultra-high net worth guide to growing and protecting assets* (pp. 55–62). Cham: Springer International Publishing. Retrieved from `https://doi.org/10.1007/978-3-030-24497-2_6`

Samuelson, P. (1969). Lifetime portfolio selection by dynamic stochastic programming. *The Review of Economics and Statistics*, *51*(3), 239-46.

Tobin, J. (1958). Liquidity preference as behavior towards risk. *Review of Economic Studies*, *25*(2), 65-86.

Yaari, M. E. (1965). Uncertain Lifetime, Life Insurance, and the Theory of the Consumer. *Review of Economic Studies*, *32*(2), 137-150.