Prediction of the Cedent's Loss Reserve Under Some Reinsurance Treaties

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Abstract

This article examines the impact of some traditional reinsurance treaties on the loss reserve of the ceding company. More precisely, it considers quota share treaty, surplus treaty, excess-of-loss treaty, and two combinations of them. Then, it develops a theoretical foundation for predicting the cedent's loss reserve and evaluating such prediction using the mean square error of prediction. The application of our findings has been given for a car collision insurance loss portfolio. Moreover, the impact of such reinsurance treaties on the variability of the cedent's loss reserve has been investigated through a simulation study.

Keywords: Loss Reserve, Reinsurance, Mean Square Error of Prediction, Simulation

1 Introduction

Loss reserve is the amount of money set aside by insurers to reimburse policyholders' developed claims. Some of these claims may be settled long after the policy expired. These claims are called long-term liabilities. A long-tail liability is an insurance claim that is not settled until well beyond when a policy has expired. These claims are usually associated with claims that are incurred but not reported during a policy period. This delay may be caused by a long court case that must be settled first or a lengthy investigation by the insurer. Long-tail liability tends to be associated with medical malpractice claims, employment discrimination, and occupational disease claims. Therefore, insurance companies have to predict and hold loss reserves for such losses. In contrast, to a long-tail business, a short-tail business is an insurance business where it is known that claims will be made and settled quickly.

We may have two types of loss reserves: incurred but not reported, say IBNR, and reported but not settled, say RBNS. The total loss reserves have been predicted by adding the IBNR's and RBNS's predictions. Prediction of loss reserves is an important and challenging problem for both insurers and reinsurers. The method of predicting the loss reserve depends on the insurance company's loss pattern, and it cannot be said that a specific reserve method provides the best estimate of the reserve for all insurance companies. An appropriate prediction of the loss reserves may help insurance/reinsurance companies in different directions, e.g., improving pricing methods, choosing a reinsurance policy, etc.

Since under a given reinsurance treaty, the reinsurer is also responsible, concerning some part of total loss reserves, most regulatory frameworks (including Solvency II) and accounting standards (such as the International Financial Reporting Standard 17, known as IFRS 17) require insurers and reinsurers to predict their outstanding claim for each line of business. This requirement can be understood in the sense of an existing reinsurance contract, the outstanding claim for both the cedent and reinsurer has to be predicted, separately. For more details, see England et al. (2019), Winkler and Kansal (2020), and Margraf et al. (2018), among others.

Regardless of the existence of the reinsurance treaty, there is very rich literature on the prediction of the total loss reserves. For instance, Mack (1993), Arjas (1989), and Norberg (1993, 1999), among others, based on total run-off triangles, developed some prediction methods for the total loss reserves. Antonio and Plat (2014) used detailed information on the claim's occurrence time, the delay between occurrence and reporting, the payments' occurrences and their sizes, and the final settlement to calibrate a model to historical data and predict future developed claims. Verrall et al. (2010) predicted the RBNS and the IBNR claims using claim amounts and claim counts. Martinez-Miranda et al. (2012) extended Verrall et al. (2010)'s model by introducing a double chain ladder, say DCL, model. Martinez-Miranda et al. (2015) developed the DCL model for situations where some prior knowledge of the number of zero-claims and the relationship between the developed claims is available. Other authors such as Salzmann and Wüthrich (2012), Merz et al. (2014), Crevecoeur et al. (2019), Duval and Pigeon (2019), Maciak et al. (2018), Noviyanti et al. (2019), Baudry and Robert (2017), Wüthrich (2018), among others, considered a variety of methods, including multivariate models, copula models, machine learning approaches, and neural network approaches.

The impact of a reinsurance treaty on many actuarial aspects of an insurance company has been studied by many authors. For instance, Kasozi et al. (2013) indicated that quota share reinsurance does have a positive impact on the survival of insurance companies as it minimizes their ultimate ruin probabilities. Riegle (2015) used the chain ladder method to predict price uncertainty under a long-tail quota share reinsurance treaty. The impact of reinsurance treaties on the insurer's lifetime has been investigated by Fan et al. (2017).

As far as we know, a small amount of literature has studied the impact of reinsurance contracts on outstanding claims. Taylor's study (1982) was the first work that predicted the outstanding claims of an insurance portfolio, under an excess-of-loss reinsurance treaty. Hertig (1985) derived a prediction for ultimate claims and current IBNR reserves under some long-term reinsurance treaties and some mild assumptions on loss ratio distribution. Craighead (1994) is considered a reinsurer that has accepted several reinsurance treaties which have given rise to catastrophe losses. Under the assumption that such catastrophe losses follow a normal development pattern, he predicted the reinsurer's gross losses using two approaches (exposure totals and statistical modeling approaches). Murphy and McLennan (2006) estimated the uncertainty of individual large claims in an insurance portfolio. They implemented a stochastic chain ladder (CL) method to project the individual large claims whenever the simulated CL factors are sampled from the observed CL factors in historical large claims. Veprauskaite and Adams (2017) studied the relationship between loss reserving errors, leverage, and reinsurance in the UK's property-casualty insurance industry. They observed that financially weak insurance companies usually underestimate reserves to reduce leverage, and so preempt costly regulatory scrutiny. Margraf et al. (2018) introduced the recovery payment method to predict the cedent's loss reserve under an excess-of-loss reinsurance treaty. More precisely, they constructed the reinsurance recovery run-off triangle as well as the total run-off triangle, then they used the DCL method to predict loss reserve for the cedent company. Úbeda Inés (2020) considered the RBNS claims data under some reinsurance treaties. Then, using two well-known actuarial loss reserving methods (chain ladder and generalized linear mixed models), he predicted future claims payments and the corresponding mean square error, for each party.

Following the above discussion, this article focuses on the problem of predicting the cedent's loss reserves under the quota share (say QS) treaty, surplus (say SPL) treaty, excess-of-loss (say XL) treaty, and two combinations of them (say QSSPL and QSXL treaties). Then, it develops a theoretical foundation for predicting the cedent's loss reserves and evaluating such prediction using the mean square error of prediction, say MSEP. Note that MSEP is used to check the predicted accuracy of the loss reserve. Moreover, this article examines the impact of these types of reinsurance treaties on the MSEP of loss reserves. This observation releases some commercial and regulatory aspects of reinsurance treaties, see Eden and Kahane (1988) for more details.

The rest of this article is organized as follows. Section 2 provides some preliminary knowledge about the loss reserving method and some reinsurance treaties. Theoretical foundations and main contributions are presented in Section 3. Section 4 illustrates a practical application of our findings on a real dataset and a simulation study. Conclusion and suggestions have been given in Section 5.

2 Preliminaries

Suppose that the data are available in a triangular form (see Figure 1 on Page 5, for a graphical representation). Let N_{ij}, for $i = 1, 2, \cdots, I; j = 0, 1, \cdots, I-1$, be the total number of claims that occurred in accident time i and reported to the insurance company at time $i+j$. Moreover, assume that X_{ij} is the claim amount that occurred in accident time i and fully paid before or at time $i+j$. Note that j is called the development year. Development year is the amount of time taken for the claim to develop from its accident year.

At the time $t = i+j$, the Figure 1's cells can be decomposed into two parts: (1) the upper triangle containing all observations and (2) the lower triangle which is unknown and has to be predicted, using an appropriate (in some sense) method.

To convince in presentation, let $\mathcal{N}_I$ and $\mathcal{D}_I$ represent the filtration based upon the past information at observation time I for the number of claims and its corresponding amounts, respectively. In other words,

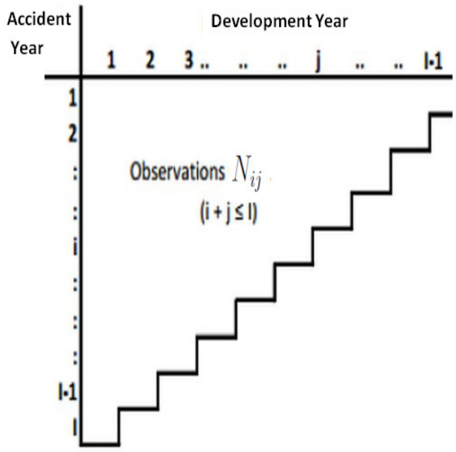
$$\mathcal{N}_I = \{N_{ij} : i = 1, \cdots, I, j = 0, \cdots, I-1; i+j \leq I\} \quad (1)$$

$$\mathcal{D}_I = \{X_{ij} : (i,j) \in \mathcal{N}_I\}. \quad (2)$$

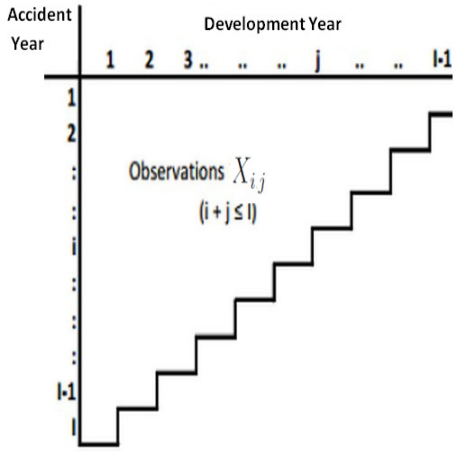
Assume that $N^{paid}_{i,j,l}$, denotes the number of future payments originating from N_{ij} claims that occurred at accident time i and were fully paid with l period delay (after being reported to the insurance company). The aggregate paid claim, denoted by N^{paid}_{ij}, has the following form

$$N^{paid}_{ij} = \sum_{l=0}^{min\{j,d\}} N^{paid}_{i,j-l,l}, \quad (3)$$

where d is the maximum delay period to pay the claim (after being reported). The total



(a)



(b)

Figure 1: IBNR tables and their corresponding run-off triangles for the number of reported claims, N_{ij}, (Panel, a) and increment payments, X_{ij}, (Panel, b).

claim amount, for each given (i, j), is denoted by X_{ij} and defined by

$$X_{ij} = \sum_{l=0}^{min\{j,d\}} \sum_{k=1}^{N^{paid}_{i,j-l,l}} Y^{(k)}_{i,j-l,l}, \tag{4}$$

where $Y^{(k)}_{i,j-l,l}$ stands for the size of the k^{th} individual claims that occurred at accident time i, reported at time $i + j - l$, and fully paid before or at time $i + j$.

In practice, some of the reported claims are settled with no payments (for example, because of fraud, franchise, etc.). So, the individual payments $Y^{(k)}_{i,j-l,l}$ can be modeled by a mixed-type distribution based on a light-tailed severity distribution (such as the Gamma distribution, for instance), with a probability mass at zero, w_i. To consider this fact, one has to consider a zero-inflated distribution for a claim severity.

Note that prior knowledge about zero-claims is considered in several studies. For example, Verrall et al. (2010) considered Gamma distribution for the density of non-zero claims. Martinez-Miranda et al. (2015) assumed non-zero claims have a distribution with a conditional mean and variance. Denuit and Trufin (2017) proposed a zero-augmented regression model for open claims. They used a mixture model with a Gamma and Pareto type 2 component augmented with a probability mass at zero.

This article employs the following zero-inflated Gamma distribution for the size of claims.

Definition 1. *A random variable $Y^{(k)}_{i,j-l,l}$ has been distributed according to the zero-inflated Gamma distribution if its density function is*

$$p_{Y^{(k)}_{i,j-l,l}}(y) = w_i I_{\{y=0\}}(y) + (1 - w_i) Gamma(\theta, \lambda) I_{\{y>0\}}(y), \tag{5}$$

where $0 \leq w_i \leq 1$ is the zero-inflation probability, and $Gamma(\theta, \lambda)$ stand for the Gamma density function with shape and scale parameters θ and λ, respectively. $I_{\{B\}}$ denotes the indicator function of event B.

The incomplete Gamma function $\Gamma(a, b)$ and the Gamma function, which plays a crucial role in the rest of this article, respectively, are defined by

$$\Gamma(a, b) = \int_0^b e^{-y} y^{a-1} dy \ \ \& \ \ \Gamma(a) = \int_0^\infty e^{-y} y^{a-1} dy, \tag{6}$$

where a and b are positive values.

As mentioned before, for the upper triangle $i+j \leq I$, total payments X_{ij} are known. But for the lower triangle $i + j > I$, the total payments X_{ij} are unknown and should be predicted. There are two types of claims: one has not been reported yet, say X^{IBNR}_{ij}, and the other one has been reported but not fully paid, say X^{RBNS}_{ij}. By taking this fact into account, total unknown payments X_{ij}, where $i + j > I$, may be decomposed at time $i + j$ as

$$\begin{aligned} X_{ij} &= \sum_{l=0}^{i+j-I-1} \sum_{k=1}^{N^{paid}_{i,j-l,l}} Y^{(k)}_{i,j-l,l} + \sum_{l=i+j-I}^{min\{j,d\}} \sum_{k=1}^{N^{paid}_{i,j-l,l}} Y^{(k)}_{i,j-l,l} \\ &= X^{IBNR}_{ij} + X^{RBNS}_{ij}. \end{aligned} \tag{7}$$

In general insurance, insurance companies seek appropriate (in some sense) reinsurance protection to reduce and homogenize their risk portfolio. A reinsurance treaty is a form of an insurance contract where a reinsurer accepts to pay a portion of an insurer's risk by receiving a reinsurance premium (Payandeh Najafabadi and Panahi Bazaz, 2018). In the reinsurance literature, the insurer is known as the first-line insurer or ceding company.

Let $Y^{(k)}, k = 1, \cdots, N^{paid}_{i,j-l,l}$, be a sequence of random claim sizes, which are independent and identically distributed, and independent of the number of claims $N^{paid}_{i,j-l,l}$. Under most reinsurance treaties, each individual claims Y are decomposed into a retention part, say Y^{In}, and a reinsured part, say Y^{Re}, where $Y = Y^{In} + Y^{Re}$ and $0 < Y^{In}, \& \ Y^{Re} < Y$.

As mentioned before, this article focuses on three well-known forms of classical reinsurance treaties and their combinations. All classical reinsurance treaties can be classified into proportional and non-proportional treaties. Under a proportional reinsurance treaty, premiums and losses share proportionally between the cedent and the reinsurance companies. Meanwhile, under a non-proportional reinsurance treaty, the reinsurer participates in those loss events that meet the reinsurance conditions.

The following provides the general concept of the QS, SPL, XL, QSXL, and QSSPL reinsurance treaties.

QS treaty: Under a quota share reinsurance treaty, the reinsurance company accepts a fixed percentage $C\%$ (where proportionality factor satisfies $0 < C < 1$) of each claim. In the other word, claim $Y^{(k)}$ shares as $Y^{(k)In-QS} = (1-C)Y^{(k)}$ and $Y^{(k)Re-QS} = CY^{(k)}$, between the cedent and the reinsurance companies, respectively.

SPL treaty: A surplus reinsurance treaty, say SPL, is a proportional reinsurance treaty which its proportionality factor depends on the sum insured (policy limit) of claim $Y^{(k)}$. Let Q_k stands for the sum insured of the claim $Y^{(k)}$. For the fixed retention line A, the cedent and the reinsurer portions, respectively, for a claim $Y^{(k)}$ are

$$Y^{(k)In-SPL} = Y^{(k)} I_{\{Q_k \leq A\}} + A\frac{Y^{(k)}}{Q_k} I_{\{Q_k > A\}} \quad and \quad Y^{(k)Re-SPL} = (1 - \frac{A}{Q_k})Y^{(k)} I_{\{Q_k > A\}}. \tag{8}$$

XL treaty: The excess-of-loss reinsurance treaty, say XL, is a non-proportional reinsurance treaty in which the reinsurer agrees to pay all claims that exceed an agreed amount R, called retention level. Therefore, for claim $Y^{(k)}$ the reinsurance company pays $Y^{(k)Re-XL} = (Y^{(k)} - R)_+$ and the cedent company pays $Y^{(k)In-XL} = \min\{Y^{(k)}, R\}$, where $(Y^{(k)} - R)_+ = \max\{Y^{(k)} - R, 0\}$.

QSXL treaty: If a reinsurer applies an excess-of-loss after a quota share, we denote such a combination by QSXL. Under the QSXL treaty, the contribution of the cedent, and the reinsurance companies are $Y^{(k)In-QSXL} = (1-C)\min\{Y^{(k)}, R\}$ and $Y^{(k)Re-QSXL} = Y^{(k)} - Y^{(k)In-QSXL}$, respectively.

QSSPL treaty: A combination of a surplus and a quota share treaties can be named by QSSPL. Under this reinsurance treaty contribution of the cedent and the reinsurance companies, respectively, are

$$Y^{(k)In-QSSPL} = (1-C)Y^{(k)} I_{\{Q_k \leq A\}} + (1-C)A\frac{Y^{(k)}}{Q_k} I_{\{Q_k > A\}}, \tag{9}$$

$$Y^{(k)Re-QSSPL} = Y^{(k)} - Y^{(k)In-QSXL}. \tag{10}$$

3 Main results

This section employs a model introduced by Verrall et al. (2010) and Martinez-Miranda et al. (2012, 2015) to predict the net loss reserve under the QS, SPL, and XL treaties, and two combinations of them. Before going into the details, we introduce the following assumptions that will be in use hereafter now.

Model Assumption 1. *Assume:*

A_1 : *The total number of claims for the accident time i and reported time j, N_{ij}, follows a Poisson distribution with mean $\alpha_i\beta_j$, where $\sum_{j=0}^{I-1}\beta_j = 1$ and N_{ij}-s are independent random variables;*

A_2 : *Conditionally on N_{ij}, the number of paid claims is distributed according to a multinomial distribution. In other words, for each given (i,j), the random vector $(N^{paid}_{i,j,0},\cdots,N^{paid}_{i,j,d}) \sim multinomial(N_{ij};p_0,\cdots,p_d)$, where delay probabilities $p_0,...,p_d$ satisfy $0 \leq p_l \leq 1$ and $\sum_{l=0}^{d} p_l = 1$;*

A_3 : *Random individual payment $Y^{(k)}_{i,j-l,l}$ has been distributed according to the zero-inflated Gamma, given by Definition (1). The cumulative distribution function, the first moment, and the variance of the zero-inflated Gamma distribution can be given by*

$$
\begin{aligned}
F_{Y^{(k)}_{i,j-l,l}}(y^{(k)}_{i,j-l,l}) &= 1-w_i+w_i\frac{\Gamma(\theta,\lambda y^{(k)}_{i,j-l,l})}{\Gamma(\theta)}, && (11)\\
E[Y^{(k)}_{i,j-l,l}] &= (1-w_i)\mu\gamma_i, && (12)\\
Var[Y^{(k)}_{i,j-l,l}] &= (1-w_i)\sigma^2\gamma_i^2+w_i(1-w_i)\mu^2\gamma_i^2; && (13)
\end{aligned}
$$

A_4 : *$Y^{(k)}_{i,j-l,l}$, are independent of N_{ij}.*

Assumption A_3 acknowledges the fact that the reported claims can be closed without a payment being made, and

$$
\begin{aligned}
E(Y^{(k)}_{i,j-l,l}|Y^{(k)}_{i,j-l,l}>0) &= \mu\gamma_i && (14)\\
Var(Y^{(k)}_{i,j-l,l}|Y^{(k)}_{i,j-l,l}>0) &= \sigma^2\gamma_i^2, && (15)
\end{aligned}
$$

where μ and σ^2 are mean and variance of an individual (non-zero) claim severity, respectively. γ_i is the inflation in the accident year i.

It is well-known that the conditional expectation of $E(X_{ij}|\mathcal{N}_I)$ plays an essential role in predicting future loss liabilities, see Wüthrich and Merz (2008) and Taylor (2012), among others for more details.

In this article, we focus on the cedent's loss reserve. So, we define the reserve at time I as $E(X^{In}_{ij}|\mathcal{N}_I)$. This term is often called "best estimate reserve" at time I. To estimate the quality of the estimated reserves, we calculate the conditional mean square error of prediction, say MSEP.

The following provides the best estimation of the cedent's loss reserve, under a general reinsurance treaty, and its corresponding conditional MSEP for the cedent's share for random claim X_{ij}.

Theorem 1. *Suppose filtration $\mathcal{N}_I$ provides all available information about the number of paid claims in a loss triangle. Moreover, besides assumptions A_1, to A_4, given in Model Assumption (1), assume $E(N^{paid}_{i,j-l,l})<\infty$, $E(Y^{(k)}_{i,j-l,l})<\infty$, $Var(N^{paid}_{i,j-l,l})<\infty$ and $Var(Y^{(k)}_{i,j-l,l})<\infty$. If X^{In}_{ij} stands for the cedent's contribution on the loss payment X_{ij}, under a given reinsurance treaty, then the best estimation for X^{In}_{ij}, and its corresponding* *conditional MSEP, respectively, are*

$$E(X_{ij}^{In}|\mathcal{N}_I) = (1-w_i)\gamma_i\mu^{In}\sum_{l=0}^{min\{j,d\}} N_{i,j-l}p_l, \tag{16}$$

$$MSEP_{\mathcal{N}_I}(X_{ij}^{In},\hat{X}_{ij}^{In}) \approx (1-w_i)\gamma_i^2(\mu^{2In}+\sigma^{2In})\sum_{l=0}^{min\{j,d\}} N_{i,j-l}p_l. \tag{17}$$

Proof. By conditioning on $N_{i,j-l,l}^{paid}$, one may conclude that

$$\begin{aligned}
E(X_{ij}^{In}|\mathcal{N}_I) &= E(E(X_{ij}^{In}|N_{i,j-l,l}^{paid})|\mathcal{N}_I)\\
&= E(E(\sum_{l=0}^{min\{j,d\}}\sum_{k=1}^{N_{i,j-l,l}^{paid}} Y_{i,j-l,l}^{(k)In}|N_{i,j-l,l}^{paid})|\mathcal{N}_I)\\
&= \sum_{l=0}^{min\{j,d\}} E(N_{i,j-l,l}^{paid}|\mathcal{N}_I)E(Y_{i,j-l,l}^{(k)In})\\
&= (1-w_i)\gamma_i\mu^{In}\sum_{l=0}^{min\{j,d\}} N_{i,j-l}p_l,
\end{aligned} \tag{18}$$

where the last equation arrives from the facts that $E(N_{i,j-l,l}^{paid}|\mathcal{N}_I) = N_{i,j-l}p_l$ and $Y_{i,j-l,l}^{(k)In}$ stands for the cedent's contribution to the individual payment $Y_{i,j-l,l}^{(k)}$ where $E(Y_{i,j-l,l}^{(k)In}) = (1-w_i)\gamma_i\mu^{In}$ and $Var(Y_{i,j-l,l}^{(k)In}) = (1-w_i)\gamma_i^2(\sigma^{2In}+\mu^{2In})$, respectively.

The conditional MSEP can be written as

$$MSEP_{\mathcal{N}_I}(X_{ij}^{In},\hat{X}_{ij}^{In}) = Var(X_{ij}^{In}|\mathcal{N}_I) + E[(\hat{X}_{ij}^{In} - E(X_{ij}^{In}|\mathcal{N}_I))^2|\mathcal{N}_I], \tag{19}$$

where the first term is well-known as the process variance and the second term is the estimation error.

Using the fact that $\hat{X}_{ij}^{In}$ and $E(X_{ij}^{In}|\mathcal{N}_I)$ are $\mathcal{N}_I$-measurable, we may conclude that $E[(\hat{X}_{ij}^{In} - E(X_{ij}^{In}|\mathcal{N}_I))^2|\mathcal{N}_I] = (\hat{X}_{ij}^{In} - E(X_{ij}^{In}|\mathcal{N}_I))^2$. This fact along with $\hat{X}_{ij}^{In} = E(X_{ij}^{In}|\mathcal{N}_I)$, may help us to restate the above conditional MSEP as

$$MSEP_{\mathcal{N}_I}(X_{ij}^{In},\hat{X}_{ij}^{In}) = \underbrace{E(Var(X_{ij}^{In}|N_{i,j-l,l}^{paid})|\mathcal{N}_I)}_{(I)} + \underbrace{Var(E(X_{ij}^{In}|N_{i,j-l,l}^{paid})|\mathcal{N}_I)}_{(II)}. \tag{20}$$

Now observe that Part (I) can be simplified as

$$\begin{aligned}
E(Var(X_{ij}^{In}|N_{i,j-l,l}^{paid})|\mathcal{N}_I) &= E(Var(\sum_{l=0}^{min\{j,d\}}\sum_{k=1}^{N_{i,j-l,l}^{paid}} Y_{i,j-l,l}^{(k)In}|N_{i,j-l,l}^{paid})|\mathcal{N}_I)\\
&= \sum_{l=0}^{min\{j,d\}} E(N_{i,j-l,l}^{paid}|\mathcal{N}_I)Var(Y_{i,j-l,l}^{(k)In})\\
&= \sum_{l=0}^{min\{j,d\}} N_{i,j-l}p_l Var(Y_{i,j-l,l}^{(k)In}).
\end{aligned} \tag{21}$$

Similarly, Part (II) can be restated as

$$\begin{aligned} Var(E(X_{ij}^{In}|N_{i,j-l,l}^{paid})|\mathcal{N}_I) &= Var(E(\sum_{l=0}^{min\{j,d\}}\sum_{k=0}^{N_{i,j-l,l}^{paid}} Y_{i,j-l,l}^{(k)In}|N_{i,j-l,l}^{paid})|N_I) \\ &= \sum_{l=0}^{min\{j,d\}} N_{i,j-l}p_l(1-p_l)(E(Y_{i,j-l,l}^{(k)In}))^2. \end{aligned} \tag{22}$$

Substituting the above findings in Equation (20) leads to

$$\begin{aligned} Var(X_{ij}^{In}|\mathcal{N}_I) &= \sum_{l=0}^{min\{j,d\}} N_{i,j-l}p_l Var(Y_{i,j-l,l}^{(k)In}) + \sum_{l=0}^{min\{j,d\}} N_{i,j-l}p_l(1-p_l)\left[E(Y_{i,j-l,l}^{(k)In})\right]^2 \\ &= \sum_{l=0}^{min\{j,d\}} N_{i,j-l}p_l(1-w_i)\gamma_i^2(\sigma^{2In}+\mu^{2In}) + \sum_{l=0}^{min\{j,d\}} N_{i,j-l}p_l(1-p_l)\left[(1-w_i)\mu^{In}\gamma_i\right]^2 \\ &\approx (1-w_i)\gamma_i^2(\mu^{2In}+\sigma^{2In})\sum_{l=0}^{min\{j,d\}} N_{i,j-l}p_l \\ &= \gamma_i\frac{\mu^{2In}+\sigma^{2In}}{\mu^{In}}E(X_{ij}^{In}|\mathcal{N}_I) \end{aligned} \tag{23}$$

where the approximation arrives by setting $(1-p_l)(1-w_i)\mu^{2In} \approx 0$. Thus, the variance will be a proportional of the mean. This means that an over-dispersed Poisson can be used to approximate the parameters, as in Martinez-Miranda et al. (2012). □

Note that, for $i+j>I$, the RBNS and the IBNR components, respectively, can be estimated by

$$\hat{X}_{ij}^{RBNS} = \sum_{l=i+j-I}^{min\{j,d\}} N_{i,j-l}\hat{p}_l\hat{\gamma}_i\hat{\mu}^{In} \tag{24}$$

$$\hat{X}_{ij}^{IBNR} = \sum_{l=0}^{i+j-I-1} \hat{N}_{i,j-l}\hat{p}_l\hat{\gamma}_i\hat{\mu}^{In}, \tag{25}$$

Now we apply the results of Theorem (1) against the above five mentioned reinsurance treaties.

The following simplifies Theorem (1)'s result under the quota share reinsurance treaty.

Proposition 1. *Suppose all Theorem (1)'s assumptions hold and the given reinsurance treaty is a quota share with the proportionality factor C. Then, the results of Theorem (1) can be simplified as*

$$E(X_{ij}^{In-QS}|\mathcal{N}_I) = (1-C)(1-w_i)\gamma_i\mu\sum_{l=0}^{min\{j,d\}} N_{i,j-l}p_l \tag{26}$$

$$MSEP_{\mathcal{N}_I}(X_{ij}^{In-QS},\hat{X}_{ij}^{In-QS}) \approx (1-C)^2(1-w_i)\gamma_i^2(\sigma^2+\mu^2)\sum_{l=0}^{min\{j,d\}} N_{i,j-l}p_l \tag{27}$$

Proof. Substituting $X_{ij}^{In} = (1 - C)X_{ij}$ into Theorem (1)'s results lead to the desired results. □

The following simplifies the Theorem (1)'s findings under the surplus reinsurance treaty.

Proposition 2. *Suppose the given reinsurance treaty in Theorem (1) is a surplus reinsurance treaty, say SPL, with retention line A. Moreover, suppose: (1) that positive random variable Q, with density function $f_Q(\cdot)$ and cumulative distribution function $F_Q(\cdot)$, stands for the sum insured corresponding to individual random claim Y, (2) both of individual random claims $Y_1, Y_2, \cdots$ and their corresponding random sum insured $Q_1, Q_2, \cdots$ are i.i.d., and (3) the first two conditional moment of random ratio $V := Y/Q$, say loss degree, satisfy $E(V|Q = q) = \vartheta < \infty$ and $E(V^2|Q = q) = \tau^2 + \vartheta^2 < \infty$, respectively.*

Then, the results of Theorem (1) can be simplified as

$$\begin{aligned} E(X_{ij}^{In-SPL}|\mathcal{N}_I) &= (1 - w_i)\gamma_i\vartheta\left[\int_0^A qf_Q(q)dq + A(1 - F_Q(A))\right] \\ &\times \sum_{l=0}^{min\{j,d\}} N_{i,j-l}p_l \end{aligned} \tag{28}$$

$$\begin{aligned} MSEP_{\mathcal{N}_I}(X_{ij}^{In-SPL}, \hat{X}_{ij}^{In-SPL}) &\approx (1 - w_i)\gamma_i^2(\vartheta^2 + \tau^2)\left[\int_0^A q^2 f_Q(q)dq + A^2(1 - F_Q(A))\right] \\ &\times \sum_{l=0}^{min\{j,d\}} N_{i,j-l}p_l. \end{aligned} \tag{29}$$

Proof. Using the moment calculation approach recommenced by Verlaak and Beirlant (2003) and Albrecher et al. (2017), the first two moments of Equation (8) can be calculated, respectively, as the following

$$\begin{aligned} E(Y^{In-SPL}) &= \int_0^A qE(V)f_Q(q)dq + A\int_A^\infty E(V)f_Q(q)dq \\ &= \vartheta\int_0^A qf_Q(q)dq + \vartheta A\int_A^\infty f_Q(q)dq \\ &= \vartheta\left[\int_0^A qf_Q(q)dq + A(1 - F_Q(A))\right] \end{aligned} \tag{30}$$

and

$$\begin{aligned} E((Y^{In-SPL})^2) &= \int_0^A q^2E(V^2)f_Q(q)dq + A^2\int_A^\infty E(V^2)f_Q(q)dq \\ &= (\vartheta^2 + \tau^2)\left[\int_0^A q^2 f_Q(q)dq + A^2(1 - F_Q(M))\right]. \end{aligned} \tag{31}$$

These two observations complete the desired results. □

From the theoretical viewpoint, one has to assume that the parameters ϑ and τ are dependent on the given $Q = q$, but from the practical viewpoint, one may justify such an independent assumption, especially in a situation where the sums insured do not fluctuate too much across policies, see Albrecher et al., (2017) for more details.

A simplification of Theorem (1) under the excess-of-loss reinsurance treaty, say XL, is given by the following.

Proposition 3. *Suppose the given reinsurance treaty in Theorem (1) is an excess-of-loss treaty with retention level R. Then, under Theorem (1)'s assumptions, we have*

$$\begin{aligned} E(X_{ij}^{In-XL}|\mathcal{N}_I) &= (1-w_i)\gamma_i \left[\frac{\Gamma(\theta+1,\lambda R)}{\lambda\Gamma(\theta)} + Rw_i(1-\frac{\Gamma(\theta,\lambda R)}{\Gamma(\theta)})\right] \\ &\times \sum_{l=0}^{min\{j,d\}} N_{i,j-l}p_l \qquad (32) \\ MSEP_{\mathcal{N}_I}(X_{ij}^{In-XL},\hat{X}_{ij}^{In-XL}) &\approx (1-w_i)\gamma_i^2 \left[\frac{\Gamma(\theta+2,\lambda R)}{\lambda^2\Gamma(\theta)} + R^2 w_i(1-\frac{\Gamma(\theta,\lambda R)}{\Gamma(\theta)})\right] \\ &\times \sum_{l=0}^{min\{j,d\}} N_{i,j-l}p_l, \qquad (33) \end{aligned}$$

where $\Gamma(\theta+1,\lambda R)$ *and* $\Gamma(\theta,\lambda R)$ *stand for the incomplete Gamma function.*

Proof. Since under the XL treaty, the contribution of an insurance company to the individual claim $Y_{i,j-l,l}^{(k)}$ is $Y_{i,j-l,l}^{(k)In-XL} = \min\{Y_{i,j-l,l}^{(k)}, R\}$. Therefore, the expected cedent claim amount is equal to

$$\begin{aligned} E(Y_{i,j-l,l}^{(k)In-XL}) &= E(\min\{Y_{i,j-l,l}^{(k)}, R\}) \\ &= \int_0^R yf(y)dy + R(1-F(R)) \\ &= \frac{\Gamma(\theta+1,\lambda R)}{\lambda\Gamma(\theta)} + Rw_i(1-\frac{\Gamma(\theta,\lambda R)}{\Gamma(\theta)}) \qquad (34) \end{aligned}$$

where $\Gamma(a,b)$ and $\Gamma(a)$ stand for the incomplete Gamma function and the Gamma function, see Equation (6), respectively.

To obtain the desired result for the conditional MSEP, observe that such a conditional MSEP is equal to $Var(X_{ij}^{In-XL}|\mathcal{N}_I)$. Now using the Wald's identity for variance, one may have

$$\begin{aligned} Var(X_{ij}^{In-XL}|\mathcal{N}_I) &= E(Var(X_{ij}^{In-XL}|N_{i,j-l,l}^{paid})|\mathcal{N}_I) + Var(E(X_{ij}^{In-XL}|N_{i,j-l,l}^{paid})|\mathcal{N}_I) \\ &= \sum_{l=0}^{min\{j,d\}} E(N_{i,j-l,l}^{paid}|\mathcal{N}_I)Var(Y_{i,j-l,l}^{(k)In-XL}) \\ &+ \sum_{l=0}^{min\{j,d\}} Var(N_{i,j-l,l}^{paid}|N_I)\left[E(Y_{i,j-l,l}^{(k)In-XL})\right]^2 \\ &\approx (1-w_i)\gamma_i^2(\mu^{2In}+\sigma^{2In}) \sum_{l=0}^{min\{j,d\}} N_{i,j-l}p_l \qquad (35) \end{aligned}$$

From the fact that $(Y_{i,j-l,l}^{(k)In-XL})^2$ is a positive random variable along with $Y_{i,j-l,l}^{(k)In-XL} = \min\{Y_{i,j-l,l}^{(k)}, R\}$, one may observe that

$$\begin{aligned} E((Y_{i,j-l,l}^{(k)In-XL})^2) &= \int_0^R y^2 f(y)dy + R^2(1-F(R)) \\ &= \frac{\Gamma(\theta+2,\lambda R)}{\lambda^2\Gamma(\theta)} + R^2 w_i(1-\frac{\Gamma(\theta,\lambda R)}{\Gamma(\theta)}). \end{aligned} \tag{36}$$

□

The following two propositions provide the result of Theorem (1) under the QSXL and the QSSPL treaties.

Proposition 4. *Under the QSXL treaty with parameters C and R, Theorem (1) can be restated as*

$$\begin{aligned} E(X_{ij}^{In-QSXL}|\mathcal{N}_I) &= (1-C)(1-w_i)\gamma_i\left[\frac{\Gamma(\theta+1,\lambda R)}{\tau\Gamma(\theta)} + Rw_i(1-\frac{\Gamma(\theta,\lambda R)}{\Gamma(\theta)})\right] \\ &\times \sum_{l=0}^{min\{j,d\}} N_{i,j-l}p_l \end{aligned} \tag{37}$$

$$\begin{aligned} MSEP_{\mathcal{N}_I}(X_{ij}^{In-QSXL}, \hat{X}_{ij}^{In-QSXL}) &\approx (1-C)^2(1-w_i)\gamma_i^2\left[\frac{\Gamma(\theta+2,\lambda R)}{\lambda^2\Gamma(\theta)} + R^2w_i(1-\frac{\Gamma(\theta,\lambda R)}{\Gamma(\theta)})\right] \\ &\times \sum_{l=0}^{min\{j,d\}} N_{i,j-l}p_l. \end{aligned} \tag{38}$$

Proof. The desired proof arrives by the fact that $Y_{i,j-l,l}^{In-QSXL} = (1-C)Y_{i,j-l,l}^{In-XL}$. □

Proposition 5. *Under the QSSPL treaty with parameters C and A, Theorem (1) can be restated as*

$$\begin{aligned} E(X_{ij}^{In-QSSPL}|\mathcal{N}_I) &= (1-C)(1-w_i)\gamma_i\vartheta \\ &\times \left[\int_0^A q f_Q(q)dq + A(1-F_Q(A))\right]\sum_{l=0}^{min\{j,d\}} N_{i,j-l}p_l \end{aligned} \tag{39}$$

$$\begin{aligned} MSEP_{\mathcal{N}_I}(X_{ij}^{In-QSSPL}, \hat{X}_{ij}^{In-QSSPL}) &\approx (1-C)^2(1-w_i)\gamma_i^2(\vartheta^2+\tau^2) \\ &\times \left[\int_0^A q^2 f_Q(q)dq + A^2(1-F_Q(A))\right]\sum_{l=0}^{min\{j,d\}} N_{i,j-l}p_l. \end{aligned} \tag{40}$$

Proof. The desired proof arrives by the fact that $Y_{i,j-l,l}^{(k)In-QSSPL} = (1-C)Y_{i,j-l,l}^{(k)In-SPL}$. □

In a practical approach to predicting future liabilities, one has to estimate all unknown parameters using available information in $\mathcal{N}_I$ and $\mathcal{D}_I$. Before, considering any estimation method, we should note that

$$\begin{aligned} E(X_{ij}|\mathcal{N}_I) &= E(\sum_{k=1}^{N_{ij}^{paid}} Y_{ij}^{(k)}|\mathcal{N}_I) \\ &= E(E(\sum_{k=1}^{N_{ij}^{paid}} Y_{ij}^{(k)}|N_{ij}^{paid})|\mathcal{N}_I) \\ &= \mu\gamma_i(1-w_i)\sum_{l=0}^{min\{j,d\}} N_{i,j-l}p_l. \end{aligned} \tag{41}$$

Clearly, the unconditional mean of X_{ij} can be written as

$$\begin{aligned} E(X_{ij}) &= E(E(X_{ij}|\mathcal{N}_I)) \\ &= \mu\gamma_i(1-w_i)\sum_{l=0}^{min\{j,d\}} E(N_{i,j-l}|\mathcal{N}_I)p_l \\ &= \tilde{\alpha}_i\tilde{\beta}_j \end{aligned} \tag{42}$$

where $\tilde{\alpha}_i = (1-w_i)\mu\gamma_i\alpha_i$ and $\tilde{\beta}_j = \sum_{l=0}^{min\{j,d\}} \beta_{j-l}p_l$.

In this article, we consider two different run-off triangles: one for the number of reported claims N_{ij}, say count triangle (with parameters α_i and β_j) and another for the size of paid claims X_{ij}, say paid triangle (with parameters $\tilde{\alpha}_i$ and $\tilde{\beta}_j$).

Based on Martinez-Miranda et al. (2015), which developed the double chain ladder method to estimate the parameters, the following develops a procedure to estimate unknown parameters in the count triangle and the paid triangle.

Procedure 1. *Given available information in $\mathcal{N}_I$ and $\mathcal{D}_I$, one may estimate unknown parameters of the Model Assumption (1) by the following steps:*

Step 1: *Employ the standard chain ladder model against N_{ij} to estimate development factor $\hat{f}_j$, for $j = 1, 2, \cdots, I-1$; Now:*

Step 1-1: *Estimates β_j using*

$$\hat{\beta}_0 = \frac{1}{\prod_{m=1}^{I-1}\hat{f}_m} \qquad \hat{\beta}_j = \frac{\hat{f}_j - 1}{\prod_{m=j}^{I-1}\hat{f}_m}, \quad for\ j = 1, 2, \cdots, I-1; \tag{43}$$

Step 1-2: *Estimates α_i by*

$$\hat{\alpha}_i = \sum_{j=0}^{I-i} N_{ij} \prod_{j=I-i+1}^{I-1} \hat{f}_j, \quad for\ i+j > I; \tag{44}$$

Step 2: *Employ the standard chain ladder model against X_{ij} to estimate $\tilde{\beta}_j$ and $\tilde{\alpha}_i$ for $j = 1, 2, \cdots, I-1$ and $i+j < I$.*

Step 3: *Employ estimated $\hat{\beta}_j$ and $\hat{\hat{\beta}}_j$ and the following system of equations to estimate $\hat{p}_0, \cdots, \hat{p}_d$.*

$$\tilde{\beta}_j = \sum_{l=0}^{d} \beta_{j-l} p_l \quad for \quad j = 0, 1, \cdots, I-1. \tag{45}$$

Step 4: *Parameter γ_i can be estimated by $\hat{\gamma}_i = \hat{\hat{\alpha}}_i / \hat{\alpha}_i \hat{\mu}$.*

Step 5: *Set the number of non-zero payments in new run-off triangle and denote this triangle by $\mathcal{R}_I = \{R_{ij} : 1 \le i \le I, 0 \le j \le I-1; i+j \le I\}$. The variables R_{ij} have cross-classified mean $E(R_{ij}) = \alpha_i^R \beta_j^R$ for all (i,j). Estimate $\hat{\alpha}_i^R$ and $\hat{\beta}_j^R$ by the chain ladder algorithm. Then*

$$\hat{w}_i = 1 - \frac{\hat{\alpha}_i^R}{\hat{\alpha}_i}. \tag{46}$$

We should recall that all estimated $\hat{p}_l$ has to be non-negative and satisfy $\sum_{l=0}^{d} \hat{p}_l = 1$. Therefore, all negative values have to be removed and the last non-negative value has to adjust to get condition $\sum_l \hat{p}_l = 1$. Moreover, in the case that two parameters θ and λ are unknown, one may estimate them using the maximum likelihood method.

We should mention that the above procedure is just a simplified version of Martinez-Miranda et al. (2015) result after imposing assumption A_3.

4 Practical applications

This section shows how one may implement the above findings in a practical situation.

4.1 For a real data

We consider the car collision insurance loss portfolio from an Iranian private insurance company. The number of reported claims run-off triangle and the paid run-off triangle are shown in Table 1 and Table 2, respectively. The triangular data included all gross car collision claims incurred during the period. We consider 10 accident years over a time horizon of 10 years. The data consists of claims reported and settled from 20 March 2010 to 19 March 2020. There are 25296 claims in the data set. For each claim, a detailed record of the claim accident date, claim notification date, settlement date, and payment amount of each transaction is provided. Also included in the data are many characteristics of the policy, policyholder, claim, sum insured, issue date, and information about the car. We consider the claim accident time, reporting time, settlement time, payment amount, and sum insured as the key information for each claim. We distinguish between two types of transactions in terms of developing a claim. A type 1 transaction refers to the settlement of a claim without payment (zero-claims). A type 2 transaction refers to the settlement of a claim by payment. Note that due to confidentiality, the insurance

Table 1: The number of reported claims run-off triangle.

i\j	0	1	2	3	4	5	6	7	8	9
1	707	745	746	747	747	747	748	750	750	750
2	2147	2234	2235	2237	2238	2238	2274	2274	2275	
3	2640	2730	2732	2734	2737	2756	2756	2756		
4	2287	2396	2398	2400	2418	2418	2418			
5	1877	1966	1982	1988	1988	1993				
6	2918	3065	3070	3072	3076					
7	3207	3307	3312	3318						
8	3017	3083	3090							
9	2940	3047								
10	2573									

Table 2: The paid run-off triangle.

i\j	0	1	2	3	4	5	6	7	8	9
1	752	868	869	875	875	875	875	875	875	875
2	3065	3440	3440	3477	3477	3477	3477	3477	3511	
3	4938	5296	5318	5344	5360	5373	5373	5373		
4	4825	5408	5422	5429	5432	5432	5432			
5	4722	5340	5485	5504	5504	5504				
6	8958	10168	10405	10417	10417					
7	11340	12393	12432	12465						
8	11677	11793	11939							
9	13650	14537								
10	19847									

company's data is multiplied by a fixed number.

Now, we consider the following null hypothesis

H_0: Individual claims have been distributed according to the zero-inflated Gamma distribution (Definition, 1)

The p-value of the Kolmogorov-Smirnov test ($p-value = 0.1630$) fails to reject the null hypothesis at a confidence level of 95%.

After a visual investigation, we candidate the log-normal distribution for the random sum insured. To validate such a conjecture, the following null hypothesis has been considered.

H_0: Random sum insured Q has been distributed according to a Log-normal distribution

Again, the Kolmogorov-Smirnov test ($p-value = 0.0861$) fails to reject the null hypothesis at a confidence level of 95%.

Table 3: Estimated parameters.

$\hat{p}_l$	0.9390	0.0427	0.0080	0.0023	0.0003	0.0004	0.0074	0.0000	0.0000	–
$\hat{\gamma}_i$	1	0.4623	0.5884	0.6778	0.88295	1.0154	1.1249	1.1591	1.4442	2.4359
$\hat{f}_j$	1.0377	1.0017	1.0007	1.0002	1.0002	1	1	1.0003	1	–
$\hat{\tilde{f}}_j$	1.0832	1.0110	1.0032	1.0006	1.0006	1	1	1.0078	1	–
$\hat{\alpha}_i^R$	745	2227	2725.92	2383.80	1969.66	3057.64	3304.27	3073.36	3033.44	2670.06
$\hat{\alpha}_i$	750	2275	2756.91	2419.64	2003.39	3099.37	3349.80	3223.59	3086.28	2706.03
$\hat{\tilde{\alpha}}_i$	875.09	3510.76	5414.57	5474.43	5546.96	10504.83	12577.97	12085.44	14878.12	22002.08
$\hat{\beta}_j$	0.9508	0.0364	0.0020	0.0013	0.0019	0.0023	0.0045	0.0003	0.0003	0
$\hat{\tilde{\beta}}_j$	0.9020	0.0750	0.0108	0.0032	0.0006	0.0006	0	0	0.0077	0
$\hat{w}_i$	0.01	0.02	0.01	0.01	0.02	0.01	0.01	0.02	0.02	0.01

Table 4: Prediction of loss reserve under different scenarios.

i	The cedent company's contribution under reinsurance treaty									
	QS		SPL		XL		QSXL		QSSPL	
	Reserve	MSEP	Reserve	MSEP	Reserve	MSEP	Reserve	MSEP	Reserve	MSEP
3	1	3	2	17	2	10	1	3	1	4
4	3	8	5	56	6	34	3	8	3	14
5	36	132	67	874	73	529	36	132	34	219
6	85	376	156	2487	170	1504	85	376	78	622
7	133	655	246	4335	267	2621	133	655	123	1084
8	150	759	277	5023	300	3037	150	759	138	1256
9	271	1707	499	11293	541	6829	271	1707	249	2823
10	1272	13540	2346	89567	2545	54162	1272	13540	1173	22392
Total	1951	17181	3598	113653	3903	68727	1951	17182	1799	28413

Now we employ Procedure (1) and use the `DCL` package of the statistical software `R` to estimate unknown parameters, Table 3 represents such estimates. Moreover, we estimate the mean and variance of the loss degree as $\vartheta = 0.0641$ and $\tau^2 = 0.0102$, respectively. The shape and rate parameters of Gamma distribution are estimated as $\theta = 0.6856$ and $\lambda = 0.1929$, respectively. So, the estimates of the mean and variance of an individual (non-zero) claim severity are $\mu = 3.5541$ and $\sigma^2 = 18.4249$. The sum insured (policy limit), Q, has the Log-normal distribution with $mean - log = 3.7790$ and $sd - log = 0.8764$.

Also, we choose retentions $1 - C = 0.80$ (proportionality factor in QS treaty), $A = 100$ (retention line in surplus treaty), and $R = 100$ (retention level in XL treaty) as an example. Determining the optimal retentions for quota share, surplus, and excess-of-loss treaties will be considered in future research.

Table 4 represents a prediction: for future payments net of QS, SPL, XL, QSXL, and QSSPL treaties.

4.2 For a simulation study

Simulation modeling provides an important method of analysis that is easily understood. In order to evaluate the performance of our results in Section 3, we use simulation analysis. This subsection presents the simulated loss development data.

We generate synthetic data using Algorithm (1). This algorithm is designed based on assumptions A_1 to A_4. For other possible simulation methods, see Stanard (1985), Bühlmann et al. (1980), Vaughan (1998), Narayan and Warthen (2000), Schiegl (2002), Stelljes (2006), among others.

Algorithm 1: Generate a Full IBNR table.

Input: Number of IBNR's row/column I, parameters $(\alpha_i, \beta_j, p_l, \gamma_i, \theta, \tau)$ as well as distributional parameters for the single payment $Y_{ij}^{(k)}$.

Output: A full IBNR table which contains information about N_{ij}^{report}, $Y_{ij}^{(k)}$ and X_{ij}

Set $i \leftarrow 1$;
while $i \leq I$ **do**
  Use the Poisson distribution (with intensity α_i) to generate the number of claims for the accident year i, and call it N_i;
  for $j \leftarrow 0$ ***to*** $I-1$ **do**
    Using the Multinomial distribution with parameters $(N_i, \beta_0, \cdots, \beta_{I-1})$, to generate vector $(N_{i0}^{report}, \cdots, N_{ij}^{report})'$;
    for $l \leftarrow 0$ ***to*** d **do**
      Use the Multinomial distribution with parameters $(N_{ij}^{report}, p_0, \cdots, p_d)$, to generate vector $(N_{i,j-0,0}^{paid}, \cdots, N_{i,j-d,d}^{paid})'$;
      Set $N_{ij}^{paid} = \sum_{l=0}^{min(j,d)} N_{i,j-l,l}^{paid}$;
    for $k \leftarrow 1$ ***to*** N_{ij}^{paid} **do**
      Generate single payments $Y_{i,j-l,l}^{(k)}$ from the $Gamma(\theta,\ \tau)$;
      Set $X_{ij} = \sum_{k=1}^{N_{ij}^{paid}} \sum_{l=1}^{d} Y_{i,j-l,l}^{(k)}$;
  Set $i \leftarrow i+1$.

Using Algorithm (1) and `R` software, we simulated 100000 count/paid triangles. Table 5 summarizes the mean of the cedent company's loss reserve, under QS, SPL, XL, QSSPL and QSXL reinsurance treaties, respectively.

Table (5)'s results indicate that: (1) the cedent's loss reserves, under the QS treaty, are significantly smaller than the SPL and XL treaties; (2) the MSEP under the XL treaty is smaller than the SPL's MSEP, therefore, the XL treaty is more effective than the SPL treaty; (3) for all treaties, the retention level has a significant impact on the

Table 5: Summary of the net loss reserve and the MSEP under the five treaties for 100000 simulated count/paid triangles.

Treaties' parameters	Mean of the net loss reserve (MSEP) under treaty				
(C, A, R)	QS	SPL	XL	QSXL	QSSPL
$(0.20, 30, 20)$	3125(44014)	1842(23099)	3641(54895)	2913(35134)	1473(14784)
$(0.25, 50, 50)$	2930(38680)	2610(50251)	3905(68612)	2929(38593)	1957(28266)
$(0.30, 75, 75)$	2734(33695)	3220(83984)	3906(68764)	2735(33696)	2254(41152)
$(0.40, 100, 100)$	2344(24756)	3601(113721)	3906(68767)	2344(24757)	2161(40940)
$(0.50, 150, 200)$	1953(17192)	4022(159704)	3906(68769)	1953(17191)	2011(39925)

cedent's loss reserve and the MSEP; (4) under two QSSPL and QSXL treaties, insurers purchase higher reinsurance coverage. So, both the cedent's loss reserve and the MSEP are smaller than under the SPL and the XL treaties. These findings were also pointed out by Veprauskaite and Adams (2017). Note that MSEP is used to check how close predicts are to actual values. The lower the MSEP, the closer is predicted to the actual. The lower value indicates a better prediction.

Figure 2 on Page 23 illustrates the cedent's MSEP as a function of the retention level for the SPL, the XL, the QS, the QSSPL, and the QSXL treaties. It shows that under the QS treaty, by increasing the amount of proportionality factor, a large part of the risk is transferred to the reinsurer. Therefore, the cedent's MSEP is reduced. Under the SPL treaty, it is observed that the lower the level of maintenance of the cedent, the less its MSEP. As the maintenance of the cedent increases, its MSEP will also increase. Under the XL treaty, the lower the level of maintenance of the cedent company, the less it is responsible for compensation, so the less MSEP it has. The higher the level of maintenance of the cedent company, the greater the risk borne by it, so MSEP will also increase. As shown in Figure 2, the cedent's MSEP is almost constant with an increasing maintenance level. The reason for this is that we did not have many large claims.

Under the QSSPL treaty, the more the amount of the proportionality factor of the quota share treaty increases, the higher the level of the cedent's MSEP increases as the retention level of the surplus treaty increases. Because in this case, the cedent keeps more risk under the SPL treaty and will be responsible for compensation. So, it has to hold more reserves, and it will also face more uncertainty. The lower the level of maintenance of the surplus treaty, the greater the risk will be transferred to the reinsurer, and this will reduce the cedent's MSEP.

Under the QSXL treaty, as the amount proportionality factor of the quota share treaty increases, the amount of the cedent's MSEP decreases with increasing the maintenance level of the XL treaty. This is because the number of large losses in our portfolio has been low. If the portfolio includes many large claims, increasing the level of maintenance of the XL treaty can increase the cedent's MSEP.

5 Conclusion and suggestions

Reinsurance has an important role in an insurance company's solvency. It can reduce the probability of a cedent's ruin. Insurance companies should use reinsurance to reduce their risk. The type of reinsurance treaty has an important role in risk management and investment decision-making. This article considers the problem of predicting future payments whenever an insurance company uses a reinsurance treaty to manage its risks. For this purpose, three classical reinsurance treaties (i.e. QS, SPL, and XL) and two combinations of them (i.e. QSSPL and QSXL) were used in loss reserving modelling. We showed how one can predict the cedent's share of loss reserve without making a run-off triangle of recovery payments from reinsurance treaties.

Practical implementation of our findings is illustrated against real data from a car collision insurance loss portfolio and also simulated data. We found that: (1) type of reinsurance treaty impacts on the insurer's loss reserve; (2) The reinsurance retention level impacts on the variability of the loss reserve; (3) The reinsurance's volume reduces the incidence of loss reserve uncertainty.

Results of this article may be extended to other reinsurance treaties. For example, insurers may want to cover only large claims in a particular portfolio. In such cases, LCR and ECOMOR reinsurance treaties can be used. Investigating the effect of large claims on a cedent's loss reserve will be considered in future research. It would be helpful to evaluate the usefulness of this approach by considering the other individual models. Moreover, it can be used to determine the optimal element of a given treaty, such as retention level.

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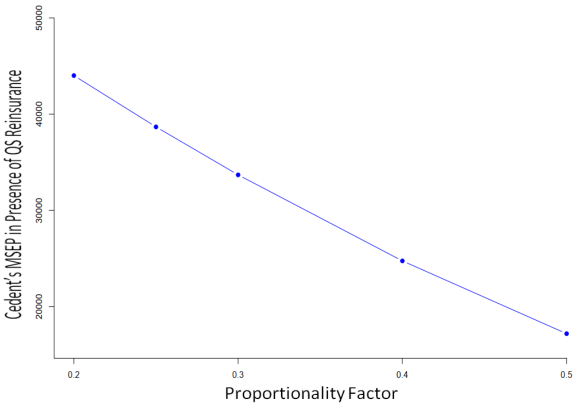
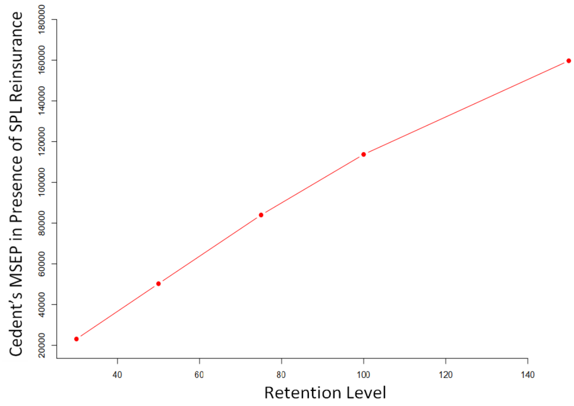


(a)



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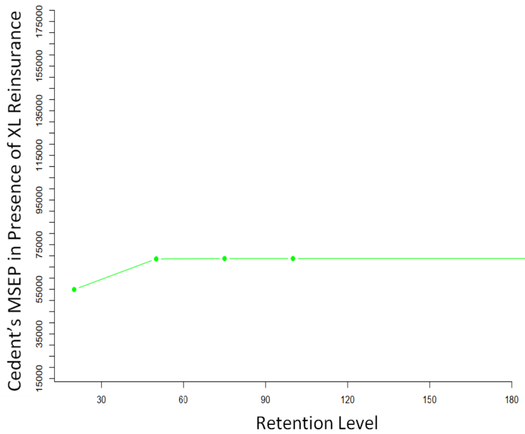

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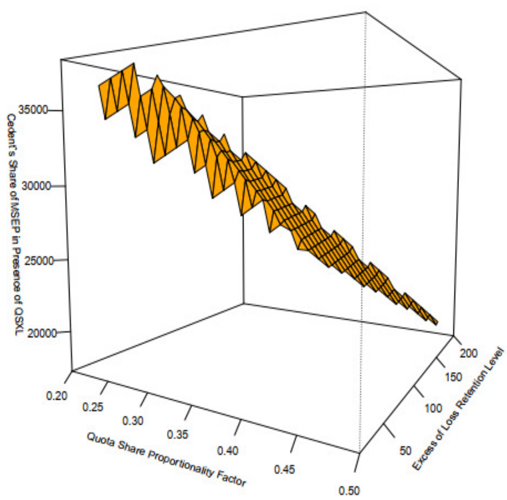

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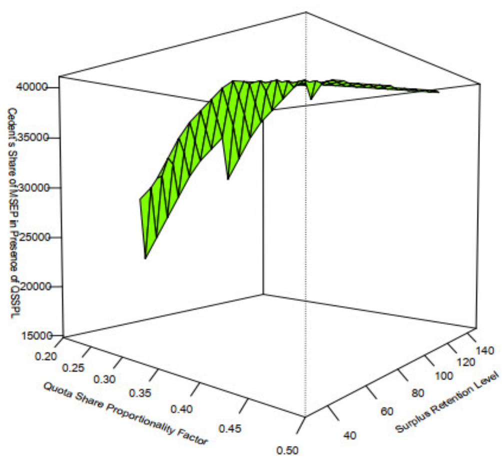

(e)

Figure 2: Panel (a) shows the confidence interval for the complete triangles, Panels (b), (c), (d), (e), and (f) show Mean of cedent's MSEP, in 100000 simulations, under QS, SPL, XL, QSXL, and QSSPL treaties, respectively.