LIFE INSURANCE MORTALITY AND LAPSATION[1]

Emiliano A. Valdez[2]
Division of Actuarial Science and Insurance
Nanyang Technological University
Singapore

Abstract

There is very little empirical evidence as to the true nature of the relationship between death and withdrawals in life insurance. This can be partly explained by the fact that applying the proper methodology offers hindrance in performing the analysis. A common assumption is that mortality and lapsation are assumed to be independent so that analysis is easier done. In this paper, we offer the method of "competing risks" in investigating the relationship between mortality and lapsation without having to assume independence. Standard actuarial models of mortality and lapsation consist of specifying distributions of the times until death and withdrawal random variables, with the minimum of the two as the only observable quantity. Typically, the two random times are assumed to be independent, however, this paper proposed a more general approach of specifying the bivariate distributions for these random times in terms of a copula function. The copula contains a parameter that measures the dependence and often includes the case of independence. We demonstrate that mispricing can occur if the possibility of antiselective lapsation is not properly accounted for.

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[1] The author wishes to thank the Nanyang Business School Research Committee for providing funding support for this research, and Mr. Raymond Cheung for providing data and programming assistance. A version of this paper was presented at the Fourth International Congress on Insurance: Mathematics and Economics in Barcelona, Spain on July 24 – 26, 2000.

[2] Dr. Emiliano A. Valdez, Ph.D, FSA, currently works in the Business School at Nanyang Technological University as Assistant Professor. He is originally from the Philippines. Dr. Valdez has also worked as an actuary for six years at Connecticut Mutual and at Price Waterhouse in U.S.A. He has published articles in the North American Actuarial Journal, the Journal of Risk and Insurance, and the Singapore International Insurance and Actuarial Journal. In 1998, Dr. Valdez was a co-winner of the David Halmstad Prize and the Edward A. Lew Award. Both awards are given annually by the Society of Actuaries to authors of papers considered to have made significant contributions to actuarial literature.

1. Introduction

Life insurance is a very complex consumer product to price. First, there are several factors that affect the cost of the product - product design, benefit level and pattern, investment returns, expenses, any guarantees, and so on. It is one unique product where its price must be determined even before the true cost of the goods sold can be assessed. As a matter of fact, it will take up to several years before the true cost can be determined. Thus, even when all the factors affecting the premium are identified, there will remain so many uncertainties in determining the level of affect each factor will have on the level of premium.

Secondly, even when all the factors are accounted for, there is the additional consideration for the possible behavior of potential buyers of the product. Some claim that purchasers of insurance behave or react differently in the presence of insurance. In the insurance market, there is often an asymmetry of information; the insurer generally does not have all the available information to accurately assess the risk level of potential policyholders. Before an insurance contract, the insurance company must assess the potential policyholder's risk through the process of underwriting. The potential insured may hold back or withhold information that can lead the insurer to reject him or her for insurance, but even if not rejected, the insurer may assess an extra premium for the presence of an exposure to "extra" hazards. *Adverse selection* is the term used to explain the "process by which prospective policyholders may gain financial advantage through insurance purchase decisions based on risk characteristics known to them, but unknown and not revealed to the insurer." See Subramanian, et al. (2000). The presence of adverse selection poses a threat to the life insurance market. This is because insurance is based on the "law of large numbers," that there is the effect of the pooling of a large number of homogeneous risks. An asymmetric information possibly creates non-homogeneous group of risks and the tendency usually is the pooling of more risks "worse" than the average risk.

Today, genetic testing poses a threat to the life and health insurance industry. This is because an individual who has undergone a genetic screening and found that he or she has a gene for a particular disease, say breast cancer, BRC1, may purchase a larger amount of insurance. The insurer may then end up with a larger pool of worse than "average" risks. The effect will even "spiral" as "good" risks realize that they are paying for the presence of more than several extra risks, thus, they lapse or withdraw their policies. The insurance, over time, will get even worse; the "bad" risks remain and the "good" risks start to leave the pool.

The relationship between mortality and lapsation is therefore of paramount importance in pricing insurance contracts and valuation of insurance liabilities. However, there is very little research done about the true nature of this relationship. The common practice is to select average mortality and lapse rates, on the "aggregate" as Jones (1997) pointed out, appropriate for a class of contract-holders. In subsequent periods, to reflect the possible mortality selection, there will be excess lapse rates for renewed policies. See also Atkinson (1990) and Dukes and MacDonald (1980). In this paper, we discuss the method of "competing risks" that is useful for analyzing the relationship. In "competing risks," we generalize the procedure by specifying the bivariate distribution of the random times in terms of a "copula" function. Expressing this distribution in terms of a copula helps avoid the problem of "identifiability" often encountered in the problems of competing risks. In the context of mortality and lapsation and in layman's term, identifiability results because the mortality pattern of those policies withdrawn cannot be observed. Once a policy is lapsed or withdrawn, it is written off the books of the insurer's business and hence, the policy cannot be traced beyond the withdrawal date. Thus it becomes difficult to demonstrate the possible difference in the mortality pattern of those individuals that lapsed and those that remain or persist. However, if the copula is specified, as shown by Carriere (1997), this problem of identifiability is eliminated.

The rest of this paper is organized as follows. We begin in section 2 with a simple illustration of how antiselection arises in life insurance. We make this demonstration using a table of decrements as is often used in actuarial pricing. In Section 3, we describe the theory of competing risks within the context of life insurance mortality and withdrawal. In this same section, we also introduce the concepts of "net" and observable "crude" lives. In section 4, we discuss the issue of identiability and how specifying the joint distribution via copulas helps resolve this issue. We discuss the procedure of estimation also in this section. In Section 4, we re-examine the issue of antiselection within the framework of competing risks. We provide concluding remarks in section 5.

2. A Simple Illustration of Antiselection

In this section, we demonstrate the consequences of antiselective lapses on the profitability of an insurer's book of business. Consider a portfolio of *n*-year term life insurance policies issued to age x. Suppose that we can partition the policyholders into two distinct classes: low risk (*L*) and high risk (*H*). The only distinguishing characteristic between these two classes is that the *L* group has lower mortality rates than the *H* group.

At issue, there will be $\ell_{x,L}$ low-risk individuals and $\ell_{x,H}$ high-risk individuals. Denote by $q^{d}_{x+t,L}$ the probability that a low-risk individual currently age $x+t$ will die within one year and $\ell_{x+t,L}$ the number of low-risk individuals active at age $x+t$. Similarly, we have $q^{d}_{x+t,H}$ the probability that a high-risk individual currently age $x+t$ will die within one year and $\ell_{x+t,H}$ the number of high-risk individuals active at age $x+t$. For lapsation, we shall use the notations $q^{w}_{x+t,L}$ and $q^{w}_{x+t,L}$ to denote the probability that a policy is lapsed or withdrawn within one year. If we let ℓ_{x+t} the total active policyholders at age $x+t$, we have

$$\ell_{x+t} = \ell_{x+t-1} - \ell_{x+t,L} \cdot \left(q^{d}_{x+t,L} + q^{w}_{x+t,L}\right) - \ell_{x+t,H} \cdot \left(q^{d}_{x+t,H} + q^{w}_{x+t,H}\right), \qquad (1)$$

for $t = 1,2,3,\ldots,n$ and with $\ell_x = \ell_{x,L} + \ell_{x,H}$, the total number of policyholders at issue. Equation (1) states that the total number of policyholders after t years consist of those that were active the previous year minus those policyholders that either died or withdrew.

Denote the benefit by B and assume that it is level all throughout the policy term. Premiums will be paid at the beginning of each year at the yearly rate of P per active policyholder and are assumed to be the same amount for both the low and high risk groups. Denote by E_{x+t} the per policy expense in year t from issue and will consist of both initial and maintenance expenses. The cashflow after t years, to be denoted by CF_t, will comprise of the premiums collected from active policyholders at the beginning of the period, $P \cdot \ell_{x+t}$, minus the benefits paid, $B \cdot v \cdot \left(\ell_{x+t,L} \cdot q^{d}_{x+t,L} + \ell_{x+t,H} \cdot q^{d}_{x+t,H}\right)$, and the expenses incurred, $E_{x+t} \cdot \ell_{x+t}$. Here, it is assumed that premiums are paid at the beginning of the year, benefits are at the end of the year, and expenses are incurred at the beginning of the year. Thus,

$$CF_t = \left(P - E_{x+t}\right) \cdot \ell_{x+t} - B \cdot v \cdot \left(\ell_{x+t,L} \cdot q^{d}_{x+t,L} + \ell_{x+t,H} \cdot q^{d}_{x+t,H}\right), \qquad (2)$$

For purposes of illustration, we shall assume that $\ell_{x,L} = 10{,}000$ and $\ell_{x,H} = 5{,}000$ policies with term $n = 10$ were issued to individuals age $x = 30$ with $B = 1{,}000$. Per policy expenses consist of 100 in the first year and yearly maintenance expense is 2 per year per policy. It is important to distinguish the large amount of first-year expense generally incurred, which is one unique aspect of a life insurance policy. This is because when a policy is withdrawn, a large amount of this expense may not be recovered. The mortality and withdrawal patterns assumed for low-risk and high-risk individuals are summarized in Table 1.

Table 1

Mortality and Withdrawal Rate Assumptions

Year	No antiselection				With antiselection			
	Low-Risk		High-Risk		Low-Risk		High-Risk	
t	$q^d_{30+t,L}$	$q^w_{30+t,L}$	$q^d_{30+t,H}$	$q^w_{30+t,H}$	$q^d_{30+t,L}$	$q^w_{30+t,L}$	$q^d_{30+t,H}$	$q^w_{30+t,H}$
0	0.0010	0.060	0.0020	0.020	0.0010	0.0667	0.0020	0.0383
1	0.0012	0.050	0.0024	0.020	0.0012	0.0580	0.0024	0.0340
2	0.0014	0.040	0.0028	0.020	0.0014	0.0493	0.0028	0.0297
3	0.0016	0.020	0.0032	0.020	0.0016	0.0306	0.0032	0.0203
4	0.0018	0.020	0.0036	0.020	0.0018	0.0320	0.0036	0.0210
5	0.0020	0.020	0.0040	0.020	0.0020	0.0333	0.0040	0.0216
6	0.0022	0.020	0.0044	0.020	0.0022	0.0346	0.0044	0.0223
7	0.0024	0.020	0.0048	0.020	0.0024	0.0359	0.0048	0.0229
8	0.0026	0.020	0.0052	0.020	0.0026	0.0372	0.0052	0.0236
9	0.0028	0.020	0.0056	0.020	0.0028	0.0385	0.0056	0.0242

Once all the components of cashflows are determined, it becomes straightforward to compute the amount of premiums. For our purposes, we will compute the gross level premium with the objective that the present value of cashflows will yield to zero. We assume no explicit provision for profits or for other contingencies. Then to quantify the effect of antiselection, we evaluate the resulting cashflows by altering the level of withdrawals. This procedure clearly makes sense because withdrawals are generally within the control of the policyholders. Thus, we consider what effect on cashflows will there be should the policyholders "consciously" select against insurer. Low-risk individuals will tend to increase their withdrawals because they feel the premium level may be too high because of some subsidy embedded to cover the extra risk from the high-risk individuals. On the other hand, high-risk individuals will tend to decrease their withdrawal rates preferring to remain in the pool of risks because they feel the premium may be a bargain to them.

Using an effective interest rate of $i = 5\%$, we present a comparison of the resulting cashflows between "no antiselection" and "with antiselection" in Table 2. The gross annual premium was set equal to $P = 39.28$ which obviously yield a zero present value of cashflow under the case of "no antiselection" but "with antiselection," this yields to a present value of -19,960 giving an indication that the antiselection results in an obvious inadequate level of premium. The premium should have been $P = 39.49$, insufficient by about 0.5%.

Table 2

Comparison of the Resulting Cashflows

		No antiselection		With antiselection	
Year t	Age x	Cashflow	Present Value	Cashflow	Present Value
0	30	-1,131,299	-1,131,299	-1,131,299	-1,131,299
1	31	306,743	292,136	307,563	292,918
2	32	254,085	230,463	254,504	230,842
3	33	208,890	180,447	208,203	179,853
4	34	171,448	141,050	169,274	139,262
5	35	136,618	107,044	133,104	104,291
6	36	104,496	77,976	99,832	74,496
7	37	75,132	53,395	69,546	49,425
8	38	48,545	32,857	42,285	28,620
9	39	24,713	15,930	18,043	11,631
Total:			0		-19,960

Figure 1 provides a graphical display of this comparison. To quantify the effect of antiselection, the assumptions in the presence of antiselection were made so that the withdrawal rates for the more risky individuals are lower than those considered low risk. By keeping the same premium with and without antiselection, we are able to demonstrate that cashflows become lower when there is the presence of antiselection. As expected, it generally causes the premium to be inadequate. This numerical illustration provides a demonstration that there is a need to adjust the assumptions so that antiselection can be properly accounted for.

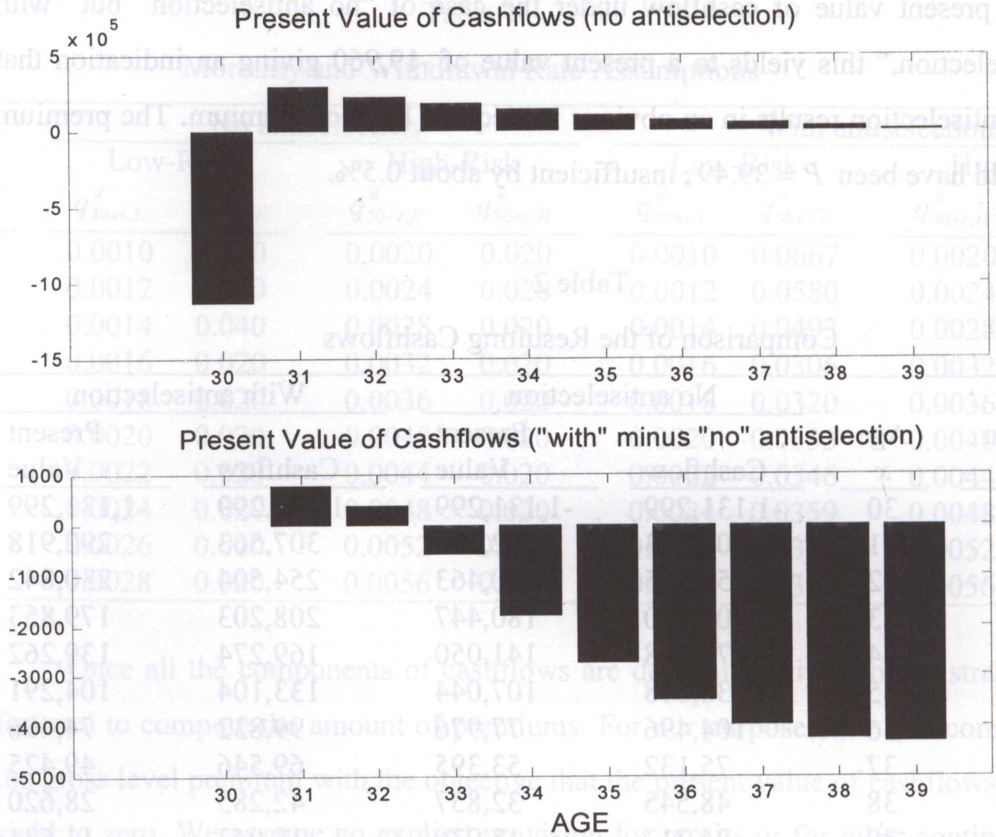

Figure 1: Comparison of the Present Value of Cashflows: "with" and "no" antiselection.

3. A Competing Risk Model of Death and Withdrawal

In this section, we provide an introduction to the subject of competing risks in the context of mortality and lapses. It does not meant to be a comprehensive treatment of the theory, however, many textbooks lay out excellent foundations of the theory. For example, see Bowers, et al. (1997), Birnbaum (1979), David and Moeschberger (1978), and Elandt-Johnson and Johnson (1980). The theory offers an alternative framework to the single decrement framework for pricing life insurance and can be used to analyze the presence of antiselective lapsation in life insurance. In actuarial science, the term "double decrement" is sometimes used to describe the competing risk framework introduced here in the context of death and withdrawals as decrements.

3.1 "Net" Lifetimes

With "competing risks," we introduce the random variables T_d and T_w as the time-until-death and the time-until-withdrawal, respectively, for a person whose age is x. These random variables are called the "net" lifetimes and we shall assume that their joint probability distribution is

$$H(t_d, t_w) = \text{Prob}(T_d \le t_d, T_w \le t_w) \tag{3}$$

and their joint survivorship function is

$$S(t_d, t_w) = \text{Prob}(T_d > t_d, T_w > t_w). \tag{4}$$

Further, we assume the joint density exists and is given by

$$h(t_d, t_w) = \partial H(t_d, t_w)/\partial t_d \partial t_w. \tag{5}$$

The "net" survivorship functions corresponding to death and withdrawal, respectively, shall be denoted by

$$S'^{(d)}(t_d) = \text{Prob}(T_d > t_d) = S(t_d, 0) \tag{6}$$

and

$$S'^{(w)}(t_w) = \text{Prob}(T_w > t_w) = S(0, t_w). \tag{7}$$

Define the "net" force of mortality by

$$\mu'^{(d)}_{x+t} = -d \log S'^{(d)}(t)/dt \tag{8}$$

and the "net" force of withdrawal by

$$\mu'^{(w)}_{x+t} = -d \log S'^{(w)}(t)/dt. \tag{9}$$

It is often customary to denote the "net" survivorship functions in (6) and (7) by

$${}_t p'^{(d)}_x = S'^{(d)}(t) \quad \text{and} \quad {}_t p'^{(w)}_x = S'^{(w)}(t),$$

with the complements also often denoted by

$${}_t q'^{(d)}_x = 1 - {}_t p'^{(d)}_x \quad \text{and} \quad {}_t q'^{(w)}_x = 1 - {}_t p'^{(w)}_x.$$

The following results are straightforward to verify:

$${}_t p'^{(d)}_x = \exp - \int_0^t \mu'^{(d)}_{x+s} ds \quad \text{and} \quad {}_t p'^{(w)}_x = \exp - \int_0^t \mu'^{(w)}_{x+s} ds.$$

Furthermore, we have

$$ {}_t q_x'^{(d)} = \int_0^t {}_s p_x'^{(d)} \mu_{x+s}'^{(d)} ds \text{ and } {}_t q_x'^{(w)} = \int_0^t {}_s p_x'^{(w)} \mu_{x+s}'^{(w)} ds. $$

Please note here that in summary, probability and survivorship functions for "net" lifetimes are denoted with a superscript prime (') such as those in equations (6) - (9).

3.2 "Overall" Lifetime

In a competing risk framework, an individual's overall lifetime is

$$ T = \min(T_d, T_w) $$

which is the observable random variable. Another random variable that is observed is the cause of failure which we shall denote by J. Thus, $J = d$ indicates that failure is due to death and $J = w$ is due to withdrawal. For purposes of simplifying results, we additionally assume that a person cannot die and withdraw simultaneously. In other words, the causes of death are mutually exclusive events. Thus, the value of J is uniquely determined with probability one and we can write this as follows $\text{Prob}(T_d = T_w) = 0$.

The joint distribution of (T, J) may be derived as follows. Consider the case of death, i.e. $J = d$. We have

$$ \begin{aligned} F_{T,J}(t,d) &= \text{Prob}(T \le t, J = d) = \text{Prob}(\min(T_d, T_w) \le t, J = d) \\ &= \text{Prob}(T_d \le t, T_d < T_w) \qquad (10) \\ &= \int_0^t \int_{t_d}^\infty h(t_d, t_w) dt_w dt_d. \end{aligned} $$

Similarly, in the case of withdrawal where $J = w$, we have

$$ \begin{aligned} F_{T,J}(t,w) &= \text{Prob}(T \le t, J = w) = \text{Prob}(\min(T_d, T_w) \le t, J = w) \\ &= \text{Prob}(T_w \le t, T_d > T_w) \qquad (11) \\ &= \int_0^t \int_{t_w}^\infty h(t_d, t_w) dt_d dt_w. \end{aligned} $$

The corresponding joint density is determined by

$$f_{T,J}(t,j) = \partial F_{T,J}(t,j)/\partial t \text{ for } j = d, w.$$

By summing the distribution functions in equations (10) and (11), we can derive the distribution function of the "overall" lifetime T. Hence,

$$F_T(t) = F_{T,J}(t,d) + F_{T,J}(t,w), \qquad (12)$$

which can be denoted by ${}_t q_x^{(\tau)}$. Alternatively, using elementary theory of probability, we can also compute this same distribution using the joint survivorship function of (T_d, T_w) as follows:

$$F_T(t) = {}_t q_x^{(\tau)} = 1 - S(t,t). \qquad (13)$$

In actuarial science, the survivorship function that corresponds to the "overall" lifetime T is customarily denoted as

$$S_T(t) = {}_t p_x^{(\tau)} = 1 - {}_t p_x^{(\tau)} = S(t,t), \qquad (14)$$

and is called the *total* or *overall* survivorship function. The "overall" force of failure can therefore be defined as

$$\mu_{x+t}^{(\tau)} = -d \log({}_t p_x^{(\tau)})/dt. \qquad (15)$$

From (15), we can easily verify that

$${}_t p_x^{(\tau)} = \exp - \int_0^t \mu_{x+s}^{(\tau)} ds. \qquad (16)$$

3.3 "Crude" Lifetimes

The survivorship functions associated with (T, J) are defined by

$$S^{(d)}(t) = \text{Prob}(T > t, J = d) \qquad (17)$$

in the case of death and

$$S^{(w)}(t) = \text{Prob}(T > t, J = w) \qquad (18)$$

in the case of withdrawal. Oftentimes, these "crude" survivorship functions are denoted respectively by ${}_t p_x^{(d)} = S^{(d)}(t)$ and ${}_t p_x^{(w)} = S^{(w)}(t)$. Similarly, the joint distribution functions in (10) and (11) are denoted by ${}_tq_x^{(d)} = F_{T,J}(t,d)$ and ${}_tq_x^{(w)} = F_{T,J}(t,w)$. From hereon, we shall these "crude" probabilities and they are often useful for estimation because they are observable from data.

Define the "crude" force of mortality by

$$\mu_{x+t}^{(d)} = -\frac{1}{p_x^{(\tau)}} \cdot \frac{d\,{}_tp_x^{(d)}}{dt} \tag{19}$$

and the "crude" force of withdrawal by

$$\mu_{x+t}^{(w)} = -\frac{1}{{}_tp_x^{(\tau)}} \cdot \frac{d\,{}_tp_x^{(w)}}{dt}. \tag{20}$$

It is straightforward to show some of the additive properties of the "crude" probabilities. For example, we have:

1. $\mu_{x+t}^{(\tau)} = \mu_{x+t}^{(d)} + \mu_{x+t}^{(w)}$;
2. ${}_tp_x^{(\tau)} = {}_tp_x^{(d)} + {}_tp_x^{(w)}$; and
3. ${}_tq_x^{(\tau)} = {}_tq_x^{(d)} + {}_tq_x^{(w)}$.

Furthermore, note that

$$\begin{aligned} {}_tp_x^{(d)} &= \text{Prob}(T > t, J = d) = \text{Prob}(\min(T_d, T_w) > t, J = d) \\ &= \text{Prob}(J = d) - \text{Prob}(T \le t, J = d) \\ &= {}_\infty q_x^{(d)} - {}_tq_x^{(d)}. \end{aligned} \tag{21}$$

Similarly, we have

$${}_tp_x^{(w)} = {}_\infty q_x^{(w)} - {}_tq_x^{(w)}. \tag{22}$$

Thus, we see that these are defective cumulative distribution functions because we have ${}_tp_x^{(d)} \ne 1 - {}_tq_x^{(d)}$ and ${}_tp_x^{(w)} \ne 1 - {}_tq_x^{(w)}$.

Corresponding to the *net*, *overall*, and *crude* probability functions, we can define life table functions. As we want to maintain the discussion of the competing risks in terms of its probabilistic structure, we will not develop the life table correspondence here in this paper. However, we suggest the reader consult Bowers, et al. (1997) and Carriere (1994).

4. The Issue of Identifiability

Because the lifetime T and the cause of failure J are the only observable quantities in competing risks, it is often important to ask how much of this information can be used to identify the rest of the probability structure. A basic issue in a competing risk problem is whether there is presence of antiselection. In the context of life insurance, this translates to examining whether individuals who withdraw their policies have generally better mortality than those that remain in the insurance pool. However, to better answer such type of question, we would need to follow-up the individuals who lapses and observe their mortality pattern. In life insurance, this is impossible to do: a policy that lapses is taken off the insurance company's book of business and hence, cannot be observed thereafter. Such is an interpretation of the issue of identifiability which is of utmost importance. As demonstrated in section 2, underpricing is a possible result when the effect of antiselective lapsation is not properly accounted in the premium calculation.

To fix ideas, suppose we have a sufficient amount of experience data to estimate the observable quantities which are the crude probabilities. Hence, the probabilities including ${}_tp_x^{(d)}$, ${}_tp_x^{(w)}$, ${}_tq_x^{(d)}$, and ${}_tq_x^{(w)}$ can be estimated from experience. We wish to know if we can determine the joint survivorship function $S(t_d, t_w)$ because knowing this will provide knowledge of the entire probability structure of the "net" lifetimes which eventually allows us to test for the presence of antiselection. First, we state and prove Tsiatis (1975) lemma. We prove it here in the context of a double decrement framework, but this result is not new and in fact, is well-known in the framework of competing risks.

From equation (17), we have

$$ {}_t p_x^{(d)} = \text{Prob}(T > t, J = d) $$
$$ = \int_t^\infty \left(\int_t^w h(t_d, t_w) dt_d \right) dt_w. $$

Now, take the derivative of both sides with respect to t, we get

$$ \frac{d}{dt} {}_t p_x^{(d)} = -\int_t^\infty h(t, t_w) dt_w. \tag{23} $$

Next, we consider

$$ S(t_d, t_w) = \int_w^\infty \int_d^\infty h(s_1, s_2) ds_1 ds_2 $$
$$ = \int_d^\infty \left(\int_w^\infty h(s_1, s_2) ds_2 \right) ds_1 $$

and take the partial derivative of both sides with respect to t_d and evaluate the term at $t_d = t_w = t$, it then follows that:

$$ \frac{\partial}{\partial t_d} S(t_d, t_w) = -\int_t^\infty h(t, s_2) ds_2. \tag{24} $$

Thus, it becomes clear that (23) and (24) are equivalent and we have the following formula made famous by Tsiatis:

$$ \frac{d}{dt} {}_t p_x^{(d)} = \frac{\partial}{\partial t_d} S(t_d, t_w). \tag{25} $$

Similarly, in the case of withdrawal when $j = w$, we have

$$ \frac{d}{dt} {}_t p_x^{(w)} = \frac{\partial}{\partial t_w} S(t_d, t_w). \tag{26} $$

Let us review the case of independence. Here, we have $S(t_d, t_w) = S'^{(d)}(t_d) \cdot S'^{(w)}(t_w)$ so that

$$\frac{\partial}{\partial t_d} S(t_d, t_w)\Big|_{t_d=t_w=t} = \left[\frac{\partial}{\partial t_d} S'^{(d)}(t_d)\right]\cdot S'^{(w)}(t_w)\Big|_{t_d=t_w=t}$$

$$= -f_d(t_d)\cdot S'^{(w)}(t_w)\Big|_{t_d=t_w=t}$$

$$= -f_d(t)\cdot S'^{(w)}(t)$$

$$= -{}_t p_x'^{(d)} \cdot \mu_{x+t}'^{(d)} \cdot {}_t p_x'^{(w)}.$$

Note that

$$\frac{d}{dt} {}_t p_x^{(d)} = -{}_t p_x^{(\tau)} \cdot \mu_{x+t}^{(d)} = -{}_t p_x'^{(d)} \cdot {}_t p_x'^{(w)} \cdot \mu_{x+t}^{(d)}.$$

From equation (25), we therefore have

$$\mu_{x+t}^{(d)} = \mu_{x+t}'^{(d)}$$

and similarly, we have

$$\mu_{x+t}^{(w)} = \mu_{x+t}'^{(w)}.$$

4.1 Using Copulas to Specify Joint Distribution

The Tsiatis formulas in equations (25) and (26) relate the "net" and "crude" survivorship functions. It is important therefore to know the probability structure of $S(t_d, t_w)$. One way to specify this probability structure is through the use of copula functions. Copulas were introduced by Sklar (1959) and today, there is an increasing amount of research on the statistical properties and applications of copulas. Some good references include Genest and MacKay (1986), Joe (1997), Frees and Valdez (1998), and Nelsen (1999).

To define a copula function more formally, we follow the definition from Schweizer and Sklar (1983). A two-dimensional copula, denoted by $C(u,v)$, is a two-dimensional probability distribution function defined on the unit square $[0,1]\times[0,1]$ and whose univariate marginals are uniform on $[0,1]$. For all u,v belonging to $[0,1]$, it is true that

$$C(u,0)=C(0,v)=0,\ C(u,1)=u, \text{ and } C(1,v)=v.$$

Other properties of distribution functions hold. Frees and Valdez (1999).

The existence of the copula for any multivariate distribution was established by Sklar (1959). He proved that for any pair of random variables, say (T_1, T_2), with a bivariate distribution function $H(t_1,t_2)=\text{Prob}(T_1 \le t_1, T_2 \le t_2)$, there will always be a copula C that will satisfy

$$H(t_1,t_2)=C(F_1(t_1),F_2(t_2)), \tag{27}$$

where $u=F_1(t_1)=\text{Prob}(T_1 \le t_1)$ and $v=F_2(t_2)=\text{Prob}(T_2 \le t_2)$ denote the marginals. Because of the result in (27), copulas are sometimes referred to as functions that link or couple the joint multivariate distribution function to their marginal distributions. We can express the result in (27) also in terms of the bivariate survivorship function by defining the survivorship copula. Let $S(t_1,t_2)=\text{Prob}(T_1 > t_1, T_2 > t_2)$ denote the bivariate survivorship function, and let

$$S_1(t_1)=\text{Prob}(T_1 > t_1)=1-F_1(t_1) \text{ and } S_2(t_2)=\text{Prob}(T_2 > t_2)=1-F_2(t_2).$$

From elementary theory of probability, we know that

$$\begin{aligned} S(t_1,t_2) &= 1-F_1(t_1)-F_2(t_2)+H(t_1,t_2) \\ &= S_1(t_1)+S_2(t_2)+H(t_1,t_2)-1 \\ &= S_1(t_1)+S_2(t_2)+C(1-S_1(t_1),1-S_2(t_2))-1. \end{aligned}$$

Thus, we see that by defining the *survivorship* copula as

$$\widetilde{C}(\widetilde{u},\widetilde{v})=\widetilde{u}+\widetilde{v}+C(1-\widetilde{u},1-\widetilde{v})-1, \tag{28}$$

we have

$$S(t_1,t_2)=\widetilde{C}(\widetilde{u},\widetilde{v}), \tag{29}$$

where $\widetilde{u}=1-u$ and $\widetilde{v}=1-v$ denote the marginal survivorship functions. For examples and further illustrations of the copula or the survivorship copula, we refer the reader to the references mentioned in the early part of this section.

4.2 Model Estimation

Suppose we have a set of n observations each of which consists of a triplet in the form $(x_i, t_i, (\delta_{i,d}, \delta_{i,w}))$ where x_i denotes the individual's age at the beginning of the observation period, or entry age (if later), $t_i = \min(t_{d_i}, t_{w_i}, t_{c_i})$ denotes the smallest of the observed times of death, withdrawal, or right-censoring, respectively, and $(\delta_{i,d}, \delta_{i,w})$ is a pair of indicator variables defined as

$$\delta_{i,d} = \begin{cases} 1, & \text{if the } i\text{th person died before the end of the observation period} \\ 0, & \text{otherwise} \end{cases}$$

and

$$\delta_{i,w} = \begin{cases} 1, & \text{if the } i\text{th person withdrew before the end of the observation period} \\ 0, & \text{otherwise} \end{cases}$$

Note that the pair (1,0) indicates death during the observation period, (0,1) indicates withdrawal during the observation period, and (0,0) indicates a right-censored observation. The pair (1,1) indicating both death and withdrawal is not possible, as we have said once withdrawal occurs, the observation is lost to follow-up. Any death information proceeding withdrawal is not obtainable.

We suggest the use of maximum likelihood procedures to estimate the parameters. Maximum likelihood estimation results in estimators with properties desirable for further statistical inference and the procedure handles well with censored observation as obviously most survival data possess. To then develop the likelihood function to maximize, we will need to distinguish the contributions made by those who were observed to die, withdraw, or survived to the end of the observation period. Without loss of generality, we can assume that the period of observation is fixed so that the right-censored time $t_{c_i} = t_c$ for all observations.

If an individual dies during the observation period, i.e. $(\delta_{i,d},\delta_{i,w})=(1,0)$, then his or her contribution to the likelihood function is given by

$$\left[\text{Prob}\left(X \le x_i + t_i, J = d \middle| X > x_i\right)\right]^{\delta_{i,d}}, \tag{30}$$

where

$$\begin{aligned}\text{Prob}\left(X \le x_i + t_i, J = d \middle| X > x_i\right) &= \frac{\text{Prob}(x_i < X \le x_i + t_i, J = d)}{\text{Prob}(X > x_i)} \\ &= \frac{\text{Prob}(x_i < X \le x_i + t_i, J = d)}{S(x_i, x_i)} \\ &= \frac{{}_{t_i}q_{x_i}^{(d)}}{S(x_i, x_i)}.\end{aligned}$$

Similarly, for an individual who withdraws or lapses during the observation period, i.e. $(\delta_{i,d},\delta_{i,w})=(0,1)$, his or her contribution to the likelihood function is given by

$$\left[\text{Prob}\left(X \le x_i + t_i, J = w \middle| X > x_i\right)\right]^{\delta_{i,w}}, \tag{31}$$

where $\text{Prob}\left(X \le x_i + t_i, J = w \middle| X > x_i\right) = \dfrac{{}_{t_i}q_{x_i}^{(w)}}{S(x_i, x_i)}$.

For an observation who survived to attain $x_i + t_i$ where $t_i = t_c$ obviously, his or her contribution to the likelihood function is given by

$$\left[\text{Prob}\left(X \le x_i + t_i \middle| X > x_i\right)\right]^{1-\delta_{i,d}-\delta_{i,w}}, \tag{32}$$

where $\text{Prob}\left(X \le x_i + t_i \middle| X > x_i\right) = \dfrac{S(x_i + t_i, x_i + t_i)}{S(x_i, x_i)}$.

By expressing the joint distribution function in terms of the copula, as expressed in (27), we can write the joint survivorship function also in terms of the copula as in (28).

The full likelihood function can be aggregated in the following manner:

$$
\begin{aligned}
L(\Omega; x_i, t_i, (\delta_{i,d}, \delta_{i,w})) &= \prod_{i=1}^{n} \left[\text{Prob}(X \le x_i + t_i, J = d | X > x_i) \right]^{\delta_{i,d}} \\
&\times \left[\text{Prob}(X \le x_i + t_i, J = w | X > x_i) \right]^{\delta_{i,w}} \qquad (33) \\
&\times \left[\text{Prob}(X \le x_i + t_i | X > x_i) \right]^{1-\delta_{i,d}-\delta_{i,w}}
\end{aligned}
$$

where Ω is the vector of parameters to estimate. Thus, the total full log-likelihood function to maximize can simplify to:

$$
\begin{aligned}
L(\Omega; x_i, t_i, (\delta_{i,d}, \delta_{i,w})) &= \sum_{i=1}^{n} \delta_{i,d} \cdot \log({}_{t_i} q_{x_i}^{(d)}) + \delta_{i,w} \cdot \log({}_{t_i} q_{x_i}^{(w)}) \\
&+ (1 - \delta_{i,d} - \delta_{i,w}) \cdot \log S(x_i + t_i, x_i + t_i) - \log S(x_i, x_i).
\end{aligned} \qquad (34)
$$

A more detailed derivation and discussion of a similar likelihood function, but in the context of dependent causes of death, in (34) can be found in Valdez (1998, 2000a). Some empirical investigation of mortality and withdrawal can also be found in Valdez (2000b).

5. Analysis of Antiselection with Competing Risks

Carriere (1998) offered a definition of antiselection in a dependent double decrement model and showed that the withdrawal benefit is smaller in the presence of antiselection. We now define what is meant by antiselection in the case of life insurance and we follow the same definition imposed by Carriere. We preserve the usual notations for time-until-death and time-until-withdrawal and their corresponding survival and distribution functions. In addition we define the force of mortality as $\mu^d(t) = f^d(t)/S^d(t)$ and the force of withdrawal as $\mu^w(t) = f^w(t)/S^w(t)$. Now consider the conditional density of T_d, given that $T_w = t$. We shall denote this by $f^{d|w}(t_d|t)$ with its corresponding survivorship function as

$$S^{d|w}(t_d|t)=\int_{t_d}^{\infty} f^{d|w}(z|t)dz$$

and the force of mortality as

$$\mu^{d|w}(t_d|t)= f^{d|w}(t_d|t)/S^{d|w}(t_d|t),$$

if this exists. It is straightforward to show that

$$S^{d|w}(t_d|t)=\exp-\int_0^{t_d} \mu^{d|w}(z|t)dz .$$

Note that in the case of independence between T_d and T_w, i.e. $h(t_d,t_w)= f^d(t_d)\cdot f^w(t_w)$, we have $f^{d|w}(t_d|t)= f^d(t_d)$, $S^{d|w}(t_d|t)= S^d(t_d)$, and $\mu^{d|w}(t_d|t)=\mu^d(t_d)$.

We are now ready to state what is meant by antiselection. We shall say that there is presence of *antiselection* at withdrawal in life insurance if the following condition holds:

$$\mu^{d|w}(t_d|t)<\mu^d(t_d) \text{ for every } t_d \geq t_w . \tag{35}$$

In other words, condition (35) states that there is a larger force of mortality after the occurrence of withdrawal. Those lives who withdrew their insurance policies tend to be generally healthier than those who kept their policies. Antiselection for life annuities can be similarly defined by simply reversing the inequality in the definition. As a consequence of this definition, it is straightforward to see that

$$S^{d|w}(t_d|t)=\exp-\int_0^{t_d}\mu^{d|w}(z|t)dz>\exp-\int_0^{t_d}\mu^d(z)dz=S^d(t_d).$$

We now provide a result which may be useful for examining the presence of antiselection. Consider the competing risk model of deaths and withdrawals with

$$H(t_d,t_w)=\text{Prob}(T_d \leq t_d,T_w \leq t_w)=C(F^d(t_d),F^w(t_w)),$$

where C is the copula function that links the univariate marginals F^d and F^w to their bivariate distribution. We shall denote the partial derivatives of the copula by

$$C_2(u,v) = \partial C/\partial v \quad \text{and} \quad C_{12}(u,v) = \partial^2 C/\partial u \partial v.$$

Therefore, we have

$$\begin{aligned} S^{d|w}(t_d|t_w) &= \int_{t_d}^{\infty} f^{d|w}(z|t_w)dz = \int_{t_d}^{\infty} \frac{h(z,t_w)}{f^w(t_w)}dz \\ &= \int_{t_d}^{\infty} \frac{f^d(z) f^w(t_w) C_{12}(F^d(z), F^w(z))}{f^w(t_w)}dz \\ &= \int_{t_d}^{\infty} f^d(z) C_{12}(F^d(z), F^w(z))dz. \end{aligned}$$

Applying a change of variable with $u = F^d(z)$, we have

$$S^{d|w}(t_d|t_w) = \int_{F^d(t_d)}^{\infty} C_{12}(u, F^w(t_w))du.$$

Applying another change of variable with $z = C_2(u, F^w(t_w))$, we have

$$\begin{aligned} S^{d|w}(t_d|t_w) &= \int_{C_2(F^d(t_d), F^w(t_w))}^{C_2(1, F^w(t_w))} dz = C_2(1, F^w(t_w)) - C_2(F^d(t_d), F^w(t_w)) \\ &= 1 C_2(F^d(t_d), F^w(t_w)). \end{aligned}$$

Note that $S^{d|w}(t_d|t) > S^d(t_d)$ if and only if

$$1 - C_2(F^d(t_d), F^w(t_w)) > 1 - F^d(t_d)$$

or equivalently

$$C_2(F^d(t_d), F^w(t_w)) < F^d(t_d) \text{ for every } t_d \geq t_w. \qquad (36)$$

Condition (36) therefore provides a condition for antiselection in terms of the copula function.

Under the case of independence, the copula function is $C(u,v)=u\cdot v$ so that

$$C_2\left(F^d(t_d),F^w(t_w)\right)=F^d(t_d).$$

Thus, condition (36) is never satisfied. It is therefore incorrect to assume independence because we always end up assuming that there is no presence of antiselection.

6. Pricing Life Insurance in a Competing Risk Framework

This section illustrates how one can use the results above to assess the price of a life insurance policy. For purposes of illustration, consider a simple n-year term life insurance that pays a death benefit of B assumed to be level and assume the policy is issued to an individual age x. Thus, the benefit ceases at age $x+n$. Suppose that there is no benefit paid upon withdrawal prior to the maturity of the policy. Assuming further a constant force of interest δ, the present value random variable in a double decrement framework as outlined in the preceding sections is given by

$$Z_{x:\overline{n|}}=\begin{cases}B\cdot e^{-\delta T}, & \text{if } 0\leq T\leq n, J=d\\ 0, & \text{if } 0\leq T\leq n, J=w\\ 0, & \text{if } T>n\end{cases} \tag{37}$$

Note that if the policyholder withdraws prior to the end of the term of the policy, no withdrawal benefit is provided. Under the actuarial equivalence principle, the net single premium for the above insurance is measured by the expectation of the random variable and the variation of this cost is measured by its variance. In the case of a policy that requires annual payment of premium, the loss random variable can then be written as

$$L=Z_{x:\overline{n|}}-\overline{P}\cdot\begin{cases}\overline{a}_{\overline{T|}}, & \text{if } 0\leq T\leq n\\ 0, & \text{if } T>n\end{cases}. \tag{38}$$

The actuarial equivalence principle computes the net annual premium $\overline{P}$ by setting the expectation of the loss random variable in (38) and thus solving $E(L)=0$. This leads us to

$$\overline{P} = B \cdot \frac{\int_0^n e^{-\delta t} dF_{T,J}(t,d)}{\int_0^n e^{-\delta t} dF_T(t)}$$

which can be approximated by

$$\overline{P} = B \cdot \frac{\sum_{k=0}^{n-1} v^{k+1} \cdot {}_k p_x^{(\tau)} \cdot q_{x+k}^{(d)}}{\sum_{k=0}^{n} v^k \cdot {}_k p_x^{(\tau)}} \tag{39}$$

where the discount rate $v = 1/(1+i) = e^{-\delta}$.

To numerically illustrate, assume that the death benefit is fixed at $B = 10{,}000$ and that effective interest rate is $i = 5\%$. Furthermore, we suppose that mortality pattern is according to the Gompertz law and that withdrawal follows an exponential distribution. We are interested in comparing the results of the net annual premium when mortality and lapses are assumed independent and when they are assumed dependent. Under independence, the copula is $C(u,v) = u \cdot v$. Under dependence, we will assume that the time-until-death and time-until-withdrawal random variables have a joint distribution expressed as a Frank's copula as follows:

$$C(u,v) = \frac{1}{\alpha} \log\left[1 + \frac{(e^{\alpha u}-1)(e^{\alpha v}-1)}{(e^{\alpha}-1)}\right].$$

See Frank (1979) and Genest (1987) for details about the statistical properties of the Frank's copula. In Figure 2, we provide the result of this comparison. The figure displays the difference in the net annual premium for varying issue ages. Note that the term of the insurance policy is up until age 65, and as expected, the net annual premium under the dependence assumption is generally larger than under independence. As the graph demonstrates, the difference in the net annual premium becomes more substantial for later issue ages. For numerical values of the net annual premiums between dependence and independence, we provide Table 3 for selected issue ages.

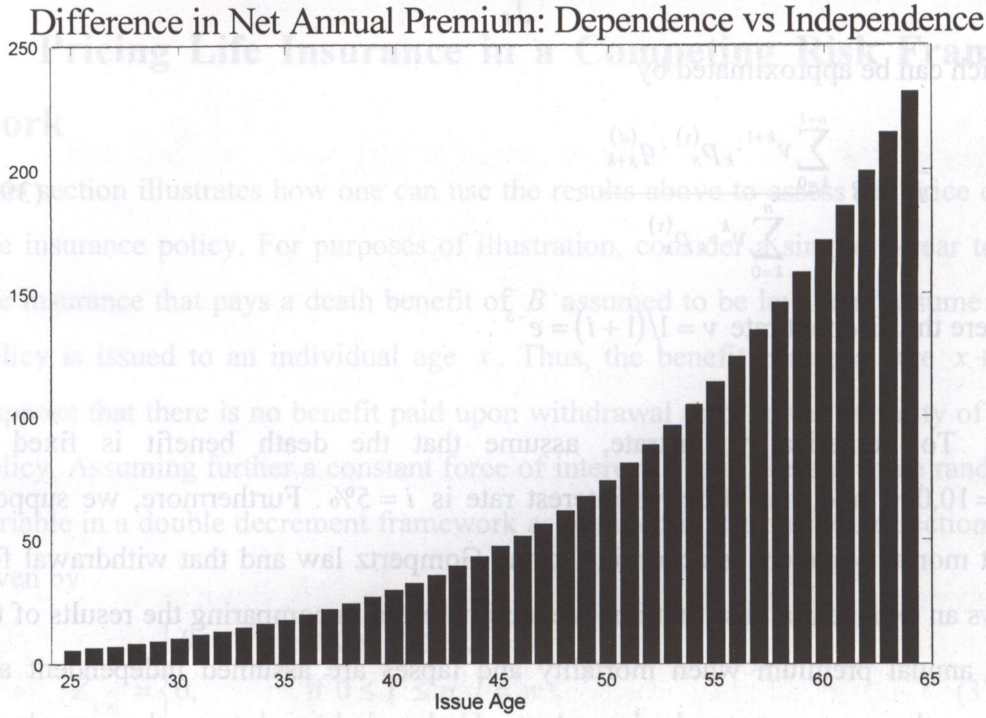

Figure 2: The Difference in the Net Annual Premium between assuming Dependence and Independence.

Table 3

The Net Annual Premium Comparison Between Dependence and Independence For Selected Issue Ages

Issue	Net Annual Premium		
Age	Dependence	Independence	Difference
25	23.90	17.80	6.10
30	33.69	22.89	10.80
35	47.62	29.45	18.17
40	67.53	37.90	29.63
45	96.09	48.80	47.29
50	136.98	62.86	74.13
55	195.01	80.98	114.03
60	275.46	104.32	171.14
64	358.76	127.68	231.08

7. Concluding Remarks

This paper explores the issue of antiselective lapsation in life insurance and offers the "competing risk" methodology as a procedure to analyze the absence or presence of antiselection and to account for its impact in pricing for life insurance products. There is antiselection if the relationship between mortality and lapsation is such that when individuals who lapse their policies are generally healthier than those who remain. This poses a threat to the industry, particularly in life insurance, when there are fewer alternatives available to the insurance company to combat antiselection especially after the policy has been issued. Premiums of life insurance are generally guaranteed at issue and there are limited clauses that can protect the insurer against antiselection. When mortality worsens as time passes, costs increase and recovery of it is difficult to achieve unless such variability has been properly accounted in pricing.

The issue of antiselection is not unique to life insurance. As a matter of fact, it is probably even more pronounced in other insurance products, except there are usually better ways to combat it. In health insurance, for example, Bluhm (1982) proposed a cumulative antiselection theory, called CAST, as a "potential explanation for the steady deterioration in loss ratios often observed on blocks of individual health insurance policies." In non-life insurance, Young (1996) explored how policyholder persistency can be factored in the credibility calculations. Young pointed out in her paper that persistency is important to consider because "the financial well-being of the insurer depends upon long-term profitability and upon spreading its risk over a large book of business." Although stated in the context of non-life insurance products, such is also true for life insurance that mortality deterioration resulting from antiselective lapsation can hurt the insurer's profitability, and hence, solvency. Insurance is the pooling of large homogeneous and predictable risks. See also Holland (1996). It is the hope of this paper that an alternative methodology is offered to study antiselection effects in life insurance.

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